

THE LAMBDA INVARIANTS AT CM POINTS

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ABSTRACT. In the paper, we show that $\lambda(z_1) - \lambda(z_2)$, $\lambda(z_1)$ and $1 - \lambda(z_1)$ are all Borcherds products in $X(2) \times X(2)$. We then use the big CM value formula of Bruinier, Kudla, and Yang to give explicit factorization formulas for the norms of $\lambda(\frac{d+\sqrt{d}}{2})$, $1 - \lambda(\frac{d+\sqrt{d}}{2})$, and $\lambda(\frac{d_1+\sqrt{d_1}}{2}) - \lambda(\frac{d_2+\sqrt{d_2}}{2})$, with the latter under the condition $(d_1, d_2) = 1$. Finally, we use these results to show that $\lambda(\frac{d+\sqrt{d}}{2})$ is always an algebraic integer and can be easily used to construct units in the ray class field of $\mathbb{Q}(\sqrt{d})$ of modulus 2. In the process, we also give explicit formulas for a whole family of local Whittaker functions, which are of independent interest.

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1. INTRODUCTION

It is well-known that the CM value $j(\tau)$ of the j -invariant at a CM point τ is always integral. What about the λ -invariants on the modular curve $X(2)$? Actually, there are six of them, satisfying the equation over $\mathbb{Q}(j)$

$$(1.1) \quad f(\lambda, j) = (1 - \lambda + \lambda^2)^3 - \frac{j}{256} \lambda^2 (1 - \lambda)^2 = 0.$$

The group $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2) = S_3$ acts on these roots simply transitively. The standard choice is

$$(1.2) \quad \lambda(\tau) = -\frac{1}{16} q^{-\frac{1}{2}} \prod_{n \geq 1} \left(\frac{1 - q^{n-\frac{1}{2}}}{1 + q^n} \right)^8, \quad q = e(\tau) = e^{2\pi i \tau}.$$

2000 *Mathematics Subject Classification.* 14G35, 14G40, 11G18, 11F27.

The first author is partially supported by a NSF grant DMS-1762289. The second author is partially supported by NSFC-11701548.

The other five roots of (1.1) are $1 - \lambda$, $\frac{1}{\lambda}$, $\frac{1}{1-\lambda}$, $\frac{\lambda-1}{\lambda}$ and $\frac{\lambda}{\lambda-1}$, all with similar product formulas. For an imaginary quadratic field $\mathbf{k} = \mathbb{Q}(\sqrt{d})$ and a positive integer N , let $\text{CM}(\mathbf{k}, N)$ be the set of CM points $\tau \in Y(N) = \Gamma(N) \backslash \mathbb{H}$ such that the associated elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ has CM by $\mathcal{O}_{\mathbf{k}}$. For every CM point $\tau \in \text{CM}(\mathbf{k}, 2)$, (1.1) implies that $\lambda(\tau)$ is integral away from 2. The following examples

$$\lambda\left(\frac{-7 + \sqrt{-7}}{2}\right) = \frac{1 + 3\sqrt{-7}}{2} \text{ and } \lambda\left(\frac{-7 + \sqrt{-7}}{4}\right) = \frac{1 + 3\sqrt{-7}}{32}$$

show that whether $\lambda(\tau)$ is integral might not be a trivial question. Our first main result indicates that the standard choice of $\lambda(\tau)$ might be the best choice for the CM point $\frac{d+\sqrt{d}}{2}$.

Theorem 1.1. *Let $d < 0$ be a fundamental discriminant of an imaginary quadratic field $\mathbf{k} = \mathbb{Q}(\sqrt{d})$. Then $\lambda_0 = \lambda(\frac{d+\sqrt{d}}{2})$ is an algebraic integer. Moreover,*

- (1) *When $d \equiv 5 \pmod{8}$, both λ_0 and $1 - \lambda_0$ are units.*
- (2) *When $d \equiv 4 \pmod{8}$, λ_0 is a unit.*
- (3) *When $d \equiv 0 \pmod{8}$, $1 - \lambda_0$ is a unit.*
- (4) *When $d \equiv 1 \pmod{8}$, $\frac{16}{\lambda_0(1-\lambda_0)}$ is a unit.*

The case $d \equiv 5 \pmod{8}$ is relatively straightforward as one can prove that $[\mathbb{Q}(\lambda_0) : \mathbb{Q}(j_0)] = 6$ with $j_0 = j(\frac{d+\sqrt{d}}{2})$. The other cases are quite subtle. When $d \not\equiv 5 \pmod{8}$, λ_0 is of degree 2 over $\mathbb{Q}(j_0)$ and its Galois conjugate may be $1 - \lambda_0$, $\frac{1}{\lambda_0}$, or $\frac{\lambda_0}{\lambda_0-1}$, but it is highly non-trivial to determine which one is its Galois conjugate in each individual case. The key technical result to prove the above theorem is

Proposition 1.2. *The Galois conjugate of λ_0 over $\mathbb{Q}(j_0)$ is $1 - \lambda_0$, $\frac{1}{\lambda_0}$ or $\frac{\lambda_0}{\lambda_0-1}$ according to $d \equiv 1, 4$ or $0 \pmod{8}$.*

The way we prove this proposition is to use the following norm formulas. Notice that Galois conjugates have to have the same norm.

Proposition 1.3. *We have the following norm formulas for $\lambda_0 = \lambda(\frac{d+\sqrt{d}}{2})$ with $d < -4$. Here h is the class number of $\mathbf{k} = \mathbb{Q}(\sqrt{d})$.*

TABLE 1. $|\text{N}(\lambda)|$

$d \pmod{8}$	0	1	4	5
$ \text{N}(\lambda_0) $	2^{2h}	2^{4h}	1	1
$ \text{N}(1 - \lambda_0) $	1	2^{4h}	2^{2h}	1
$ \text{N}(1/\lambda_0) $	2^{-2h}	2^{-4h}	1	1
$ \text{N}(\lambda_0/(\lambda_0 - 1)) $	2^{2h}	1	2^{-2h}	1

This proposition is a bi-product of the original purpose of this paper: a factorization formula for $|\text{N}(\lambda(\frac{d_1+\sqrt{d_1}}{2}) - \lambda(\frac{d_2+\sqrt{d_2}}{2}))|$ similar to the beautiful factorization formula of singular moduli of Gross and Zagier ([GZ85]).

Theorem 1.4. *Let $\mathbf{k}_j = \mathbb{Q}(\sqrt{d_j})$ with ring of integers $\mathcal{O}_{d_j} = \mathbb{Z}[\tau_j]$ with $\tau_j = \frac{d_j + \sqrt{d_j}}{2}$, $j = 1, 2$. Let w_j be the number of roots of unity in $\mathbf{k}_j$, and let $h_j = h(\mathbf{k}_j)$ be the class number of $\mathbf{k}_j$. Let $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ and $F = \mathbb{Q}(\sqrt{D})$ with $D = d_1 d_2$. For an ideal $\mathfrak{a}$ of F , let $\rho_{E/F}(\mathfrak{a})$ be the number of integral ideals of E with relative norm $\mathfrak{a}$, and let $\chi = \chi_{E/F}$ be the quadratic Hecke character of F associated to E/F . Let $c(t) = \frac{d_1^2 + d_2^2 - (d_1 + d_2)}{4} - \frac{m+D}{2} \in \mathbb{Z}$. Assume that $(d_1, d_2) = 1$. Then*

(1) *When $d_1 \equiv d_2 \equiv 5 \pmod{8}$,*

$$\begin{aligned} & \log |N(\lambda(\tau_1) - \lambda(\tau_2))| \\ &= 6 \sum_{\substack{t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F \\ |m| < \sqrt{D} \\ c(t) \in 2\mathbb{Z}}} \sum_{\chi_{E/F}(\mathfrak{p}) = -1} \left(\frac{1 + \text{ord}_{\mathfrak{p}_t}(t\mathfrak{p}_t^{-2})}{2} + \frac{\text{ord}_{\mathfrak{p}_t}(\mathfrak{p})}{3} \right) \rho_{E/F}(t\mathfrak{p}\mathfrak{p}_t^{-2}) \log N(\mathfrak{p}). \end{aligned}$$

Here $\mathfrak{p}_t$ is the prime of F above 2 with $t \in \mathfrak{p}_t$.

(2) *When $d_1 \equiv 5 \pmod{8}$, and $d_2 \equiv 0, 4 \pmod{8}$,*

$$\log |N(\lambda(\tau_1) - \lambda(\tau_2))| = \sum_{\substack{t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F \\ |m| < \sqrt{D} \\ c(t) \in 2\mathbb{Z}}} \sum_{\chi_{E/F}(\mathfrak{p}) = -1} \frac{1 + \text{ord}_{\mathfrak{p}}(t)}{2} \rho_{E/F}(t\mathfrak{p}) \log N(\mathfrak{p}).$$

(3) *When $d_1 \equiv 1 \pmod{8}$ and $d_2 \equiv 5 \pmod{8}$,*

$$\log |N(\lambda(\tau_1) - \lambda(\tau_2))| = 2 \sum_{\substack{t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F \\ |m| < \sqrt{D} \\ c(t) \in 2\mathbb{Z}}} \sum_{\chi_{E/F}(\mathfrak{p}) = -1} \frac{1 + \text{ord}_{\mathfrak{p}}(t)}{2} \rho_{E/F}(t\mathfrak{p}) \log N(\mathfrak{p}).$$

(4) *When $d_1 \equiv 1 \pmod{8}$ and $d_2 \equiv 0, 4 \pmod{8}$,*

$$\begin{aligned} & \log |N(\lambda(\tau_1) - \lambda(\tau_2))| \\ &= \frac{1}{2} \sum_{\substack{t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F \\ |m| < \sqrt{D} \\ c(t) \in 2\mathbb{Z}}} \sum_{\chi_{E/F}(\mathfrak{p}) = -1} \frac{1 + \text{ord}_{\mathfrak{p}}(t)}{2} \rho_{E/F}(t\mathfrak{p}) \log N(\mathfrak{p}) + 2^3 \frac{h_1 h_2}{w_1 w_2} \log 2. \end{aligned}$$

(5) *When $d_1 \equiv 1 \pmod{8}$ and $d_2 \equiv 1 \pmod{8}$,*

$$\begin{aligned}
& \log |N(\lambda(\tau_1) - \lambda(\tau_2))| \\
&= 2 \sum_{\substack{t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F \\ |m| < \sqrt{D} \\ c(t) \in 2\mathbb{Z}}} \sum_{\chi_{E/F}(\mathfrak{p}) = -1} \frac{1 + \text{ord}_{\mathfrak{p}}(t\mathfrak{p}_t^{-2})}{2} \rho_{E/F}(t\mathfrak{p}\mathfrak{p}_t^{-2}) \log N(\mathfrak{p}) + 2^5 \frac{h_1 h_2}{w_1 w_2} \log 2.
\end{aligned}$$

Here $\mathfrak{p}_t$ is the prime of F above 2 with $t \in \mathfrak{p}_t$.

We remark that for each $t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F$ in the above formula, the second sum $\sum_{\chi_{E/F}(\mathfrak{p}) = -1}$ has at most one non-zero term because it is impossible to have two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of F inert in E/F with both $\rho_{E/F}(\mathfrak{a}\mathfrak{p}_1)$ and $\rho_{E/F}(\mathfrak{a}\mathfrak{p}_2)$ non-zero for any fractional ideal $\mathfrak{a}$. In [Yu], one of the authors (Yu) extends the factorization formula to Rosenhain invariants on Siegel threefold.

Corollary 1.5. *Let the notation be as above.*

- (1) $N(\lambda(\frac{d_1+\sqrt{d_1}}{2}) - \lambda(\frac{d_2+\sqrt{d_2}}{2}))$ is an integer.
- (2) If $p \mid N(\lambda(\frac{d_1+\sqrt{d_1}}{2}) - \lambda(\frac{d_2+\sqrt{d_2}}{2}))$, then $p \leq \frac{D}{4}$ and $p \mid \frac{D-m^2}{4}$ for some integer $0 \leq m < \sqrt{D}$.
- (3) When $E/\mathbb{Q}$ is unramified at 2, $N(\lambda(\frac{d_1+\sqrt{d_1}}{2}) - \lambda(\frac{d_2+\sqrt{d_2}}{2}))$ is a square up to sign.

Although $N(j(\tau_1) - j(\tau_2))$ does not depend on the choice of $\tau_j \in \text{CM}(\mathbf{k}_j, 1)$, the same is NOT true for $N(\lambda(\tau_1) - \lambda(\tau_2))$. Let

$$\mathcal{N}(d_1, d_2) = \{ |N(\lambda(\tau_1) - \lambda(\tau_2))| : \tau_j \in \text{CM}(\mathbf{k}_j, 2) \}.$$

Then a simple group theoretic argument using Lemma 2.1 shows that $\mathcal{N}(d_1, d_2)$ has order at most 9. We find the following surprising result.

Theorem 1.6. *Assume $d_1, d_2 < -4$. Then the order of $\mathcal{N}(d_1, d_2)$ is given by*

$$|\mathcal{N}(d_1, d_2)| = \begin{cases} 1 & \text{if } d_1 \equiv d_2 \equiv 5 \pmod{8}, \\ 2 & \text{if exactly one of } d_1, d_2 \equiv 5 \pmod{8}, \\ 4 & \text{if } d_1 \equiv d_2 \equiv 1 \pmod{8}, \\ 5 & \text{otherwise.} \end{cases}$$

The proofs of Theorem 1.4 and Proposition 1.3 are similar to that in [YY] although they are technically more subtle and raise some interesting technical questions. In Section 2 we review and prove some basic facts about CM points in $X(2)$, λ -invariants, and ray class group actions on the CM cycles. After that we prove Theorem 1.1 and Propositions 1.2 assuming Proposition 1.3. In Section 3 we treat the product $X(2) \times X(2)$ of modular curves as the Shimura variety associated to $L = M_2(\mathbb{Z})$ with $Q(x) = 2 \det x$, and find a weakly holomorphic modular form f whose Borcherds product lifting is $\lambda(z_1) - \lambda(z_2)$ (Proposition 3.5). In the process, we also prove that $\lambda(z_1)$ and $1 - \lambda(z_1)$ are Borcherds products for some explicit weakly homomorphic forms (Corollary 3.4). Notice that the

existence of f is not guaranteed by Bruinier's converse theorem ([Bru14]). In Section 4 we use the big CM value formula in [BKY12] to compute the absolute value of the norm of $\lambda(\tau_1) - \lambda(\tau_2)$ and to prove Theorem 1.4 and Proposition 1.3, although the big CM cycle and Galois orbit are not always the same in this case. We then use these results to prove Theorem 1.6.

The main technical part to prove Theorem 1.4 and Proposition 1.3 is to compute the local Whittaker functions at primes dividing 2, which needs more general explicit formulas for local Whittaker functions than exists in literature. In Section 5 we give an explicit formula for a whole family of local Whittaker functions which are basic and of independent interest. In the last section, we actually compute some examples of the factorization of the norm $|N(\lambda(\tau_1) - \lambda(\tau_2))|$ using Theorem 1.4. We check the formulas with computer computation results, and they match perfectly.

2. THE INTEGRALITY OF THE CM VALUES OF THE LAMBDA FUNCTION λ

Let $Y(2) = \Gamma(2) \backslash \mathbb{H}$ and $X(2) = Y(2) \cup \{0, 1, \infty\}$. Then $X(2)$ is the compactification of $Y(2)$ and is known to have a canonical model over $\mathbb{Q}$. Then the λ -invariant defined by (1.2) gives an isomorphism between $X(2)$ and $\mathbb{P}^1$. Moreover, we have the following commutative diagram

$$(2.1) \quad \begin{array}{ccc} X(2) & \xrightarrow{\lambda} & \mathbb{P}^1 \\ \downarrow \pi & & \downarrow \pi' \\ X(1) & \xrightarrow{j} & \mathbb{P}^1 \end{array}$$

where π is the natural projection and π' is given by

$$\pi'(x) = \frac{256(1 - x + x^2)^3}{x^2(1 - x)^2}.$$

So the minimal polynomial of $\lambda = \lambda(\tau)$ over $\mathbb{Q}(j(\tau))$ is generically given by (1.1). For any $\tau \in \mathbb{H}$, the preimage of $j(\tau)$ under π is

$$\begin{aligned} & \left\{ \lambda(\tau), 1 - \lambda(\tau), \frac{1}{\lambda(\tau)}, \frac{1}{1 - \lambda(\tau)}, \frac{\lambda(\tau) - 1}{\lambda(\tau)}, \frac{\lambda(\tau)}{\lambda(\tau) - 1} \right\} \\ &= \{ \lambda \circ \gamma : \gamma \in \mathrm{SL}_2(\mathbb{Z}) / \Gamma(2) \} \end{aligned}$$

Let $\mathbf{k} = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with fundamental discriminant d viewed as a subfield of $\mathbb{C}$ in the usual way. For $N = 1, 2$, let $\mathrm{CM}(\mathbf{k}, N)$ be the set of CM points in $X(N)$ defined as in the introduction. They are parameterized by triples $(\mathfrak{a}, \alpha, \beta)$, where $\mathfrak{a}$ is a fractional ideal of $\mathbf{k}$, and (α, β) is an ordered $\mathbb{Z}$ -basis of $\mathfrak{a}$ such that $\frac{\alpha}{\beta} \in \mathbb{H}$. Two such triples $(\mathfrak{a}_i, \alpha_i, \beta_i)$ are equivalent if there is $z \in \mathbf{k}$ and $\gamma \in \Gamma(N)$ such that

$$\mathfrak{a}_1 = z\mathfrak{a}_2, \quad \gamma(\alpha_1, \beta_1)^t = (z\alpha_2, z\beta_2)^t.$$

The associated CM point in $X(N)$ is given by $\tau_{(\mathfrak{a}, \alpha, \beta)} = \frac{\alpha}{\beta}$. The ray class group $\text{Cl}(\mathfrak{k}, N)$ acts on $\text{CM}(\mathfrak{k}, N)$ as follows: For a fractional ideal $\mathfrak{b}$ prime to N ,

$$[\mathfrak{b}] * (\mathfrak{a}, \alpha, \beta) = (\mathfrak{a}\mathfrak{b}, \alpha_1, \beta_1)$$

with

$$\alpha_1 \equiv \alpha \pmod{N}, \text{ and } \beta_1 \equiv \beta \pmod{N}.$$

We refer to [Yan16] for more detail. Our special case $N = 1, 2$ simplifies the description a little. Although it is well-known that $\text{Cl}(\mathfrak{k}) = \text{Cl}(\mathfrak{k}, 1)$ acts on $\text{CM}(\mathfrak{k}) = \text{CM}(\mathfrak{k}, 1)$ simply transitively, it is not true even for $\text{Cl}(\mathfrak{k}, 2)$. Let $h(\mathfrak{k}, N)$ be the ray class number of $\mathfrak{k}$ of modulus N , and $h(\mathfrak{k}) = h(\mathfrak{k}, 1)$.

Lemma 2.1. *Let the notation be as above, and let $w = |\mathcal{O}_{\mathfrak{k}}^\times|$ be the number of roots of unity in $\mathfrak{k}$. Then*

(1) *One has*

$$\frac{h(\mathfrak{k}, 2)}{h(\mathfrak{k})} = \frac{2}{w} r(d) = \begin{cases} \frac{2}{w} & \text{if } d \equiv 1 \pmod{8}, \\ \frac{4}{w} & \text{if } d \equiv 0 \pmod{4}, \\ \frac{6}{w} & \text{if } d \equiv 5 \pmod{8}. \end{cases}$$

We denote $r(d) = 1, 2, 3$ according to $d \equiv 1 \pmod{8}, 0 \pmod{4}$, or $5 \pmod{8}$.

- (2) *The CM set $\text{CM}(\mathfrak{k}, 2)$ has $6/r(d)$ orbits under the action of $\text{Cl}(\mathfrak{k}, 2)$ when $d < -4$ and has 3 orbits when $d = -3, -4$.*
- (3) *When $4|d$, $\lambda(\tau)$ is real when $\tau \in \text{CM}(\mathfrak{k}, 2)$ and the minimal polynomial of $\lambda(\tau)$ over $\mathbb{Q}$ is of degree $h(\mathfrak{k}, 2)$.*
- (4) *When $4 \nmid d$, $\lambda(\tau)$ is not real for $\tau \in \text{CM}(\mathfrak{k}, 2)$, and the minimal polynomials of $\lambda(\tau)$ and $\lambda(-\bar{\tau}) = \overline{\lambda(\tau)}$ over $\mathbb{Q}$ are the same and of degree $2h(\mathfrak{k}, 2)$. In particular, when $d \equiv 5 \pmod{8}$ and $d \neq -3$, the minimal polynomial of $\lambda(\tau)$ is given by*

$$f_\lambda(x) = \prod_{z \in \text{CM}(\mathfrak{k}, 2)} (x - \lambda(z)).$$

Proof. (sketch) Claim (1) follows from the general relation between ideal class groups and ray class groups, see for example [Mil] Theorem 1.7, p. 146]:

$$\frac{h(\mathfrak{k}, \mathfrak{m})}{h(\mathfrak{k})} = \frac{N(\mathfrak{m})}{[\mathcal{O}_{\mathfrak{k}}^\times : U_{\mathfrak{m}}]} \prod_{\mathfrak{p}|\mathfrak{m}} (1 - N(\mathfrak{p})^{-1}),$$

where $\mathfrak{m}$ is an integral ideal of $\mathfrak{k}$, $h(\mathfrak{k}, \mathfrak{m})$ is the ray class number of $\mathfrak{k}$ of modulus $\mathfrak{m}$, and $U_{\mathfrak{m}}$ is the subgroup of $\mathcal{O}_{\mathfrak{k}}^\times$ consisting of units which are $1 \pmod{\mathfrak{m}}$.

Since the projection map from $X(2)$ to $X(1)$ has degree 6 and it does not ramify at the CM points unless $d = -3, -4$, claim (2) follows from (1) immediately.

For (3), we may assume $\tau = \frac{-d+\sqrt{d}}{2} = [\mathcal{O}_{\mathfrak{k}}, \alpha, 1]$ with $\alpha = \frac{-d+\sqrt{d}}{2}$. Then $-\bar{\tau} = \frac{d+\sqrt{d}}{2} = [\mathcal{O}_{\mathfrak{k}}, -\bar{\alpha}, 1]$. When $4|d$, it is clear $-\bar{\tau} = d + \tau$ which is equal to τ in $X(2)$. Now assume that d is odd. If τ and $-\bar{\tau}$ are in the same orbit, then there is an ideal $\mathfrak{b}$ such that

$$[\mathcal{O}_{\mathfrak{k}}, -\bar{\alpha}, 1] = [\mathfrak{b}] * [\mathcal{O}_{\mathfrak{k}}, \alpha, 1] = [\mathfrak{b}, \alpha_1, \beta_1]$$

So $\mathfrak{b} = z\mathcal{O}_{\mathbf{k}}$ and there is $\gamma \in \Gamma(2)$ such that

$$\gamma(\alpha_1, \beta_1)^t = (-z\bar{\alpha}, z)^t.$$

Notice that $\alpha_1 \equiv \alpha \pmod{2}$ and $\beta_1 \equiv 1 \pmod{2}$ by definition. A simple calculation shows that $\alpha + \bar{\alpha} = -d \equiv 0 \pmod{2}$, a contradiction.

(4) follows from (3). $\square$

The cases $d = -3, -4$ are special, so we record them as a separate lemma.

Lemma 2.2. *One has for $d = -3, -4$*

(1)

$$\begin{aligned} h(\mathbb{Q}(\sqrt{-4}), 2) &= h(\mathbb{Q}(\sqrt{-4})) = 1, & \lambda(\sqrt{-1}) &= -1, \\ h(\mathbb{Q}(\sqrt{-3}), 2) &= h(\mathbb{Q}(\sqrt{-3})) = 1, & \lambda\left(\frac{-3 + \sqrt{-3}}{2}\right) &= \frac{1 + \sqrt{-3}}{2}. \end{aligned}$$

(2) $\text{CM}(\mathbf{k}, 2)$ has three $\text{Cl}(\mathbf{k}, 2)$ -orbits, and $X(2) \rightarrow X(1)$ is ramified at the CM points $\sqrt{-1}$ and $\frac{-3 + \sqrt{-3}}{2}$ with ramification index 2.

Proposition 2.3. *Let $d < -4$, and let $\tau = \tau_{[\mathfrak{a}, \alpha, \beta]}$ be a CM point in $X(2)$. Then the polynomial in λ $f(\lambda, j(\tau))$ in [\(1.1\)](#) is irreducible over $\mathbb{Q}(j(\tau))$ when $d \equiv 5 \pmod{8}$ and factorizes into product of three quadratic polynomials over $\mathbb{Q}(j(\tau))$ when $d \not\equiv 5 \pmod{8}$. Equivalently,*

$$[\mathbb{Q}(\lambda(\tau)) : \mathbb{Q}(j(\tau))] = \begin{cases} 6 & \text{if } d \equiv 5 \pmod{8}, \\ 2 & \text{otherwise.} \end{cases}$$

Moreover, when $d \not\equiv 5 \pmod{8}$, we have

$$f(\lambda, j(\tau)) = f_1(\lambda)f_2(\lambda)f_3(\lambda)$$

with

$$\begin{aligned} f_1(\lambda) &= \lambda^2 - \lambda + a, \\ f_2(\lambda) &= \lambda^2 - \frac{1}{a}\lambda + \frac{1}{a}, \\ f_3(\lambda) &= \lambda^2 - \left(2 - \frac{1}{a}\right)\lambda + 1, \end{aligned}$$

for some $a \in \mathbb{Q}(j(\tau))$.

Proof. (sketch) When $4 \nmid d$, $[\mathbb{Q}(\lambda(\tau)) : \mathbb{Q}] = 2h(\mathbf{k}, 2)$ while $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h$, and so

$$[\mathbb{Q}(\lambda(\tau)) : \mathbb{Q}(j(\tau))] = 2h(\mathbf{k}, 2)/h,$$

which is 6 or 2 depending when $d \equiv 5$ or $1 \pmod{8}$ by Lemma [2.1](#). Similarly when $4 \mid d$, $[\mathbb{Q}(\lambda(\tau)) : \mathbb{Q}(j(\tau))] = h(\mathbf{k}, 2)/h = 2$.

Now assume $d \not\equiv 5 \pmod{8}$. Let $F = \mathbb{Q}(j(\tau))$. Since $F(\lambda(\tau))/F$ is of degree 2, the Galois conjugate of $\lambda(\tau)$ is either $1 - \lambda(\tau)$, $\frac{1}{\lambda(\tau)}$ or $\frac{\lambda(\tau)}{\lambda(\tau)-1}$.

First assume that the Galois conjugate of $\lambda(\tau)$ is $1 - \lambda(\tau)$, then $a = \lambda(\tau)(1 - \lambda(\tau)) \in F$, and the minimal polynomial of $\lambda(\tau)$ is $f_1(\lambda)$. At the same time, the Galois conjugate of $\frac{1}{\lambda}$

becomes $\frac{1}{1-\lambda}$, and so the minimal polynomial of $\frac{1}{\lambda(\tau)}$ is $f_2(\lambda)$. Finally, the Galois conjugate of $\frac{\lambda(\tau)}{\lambda(\tau)-1}$ becomes $\frac{\lambda(\tau)-1}{\lambda(\tau)}$, and their minimal polynomial is $f_3(\lambda)$. As the roots of $f(\lambda, j(\tau))$ are these six numbers, we see $f(\lambda) = f_1 f_2 f_3$ as claimed. The other cases are similar. $\square$

For the rest of this section, we assume Proposition [1.3](#) and use it to prove Proposition [1.2](#) and Theorem [1.1](#). Proposition [1.2](#) follows from Propositions [1.3](#) and [2.3](#) immediately as Galois conjugates have to have the same norm. Let $H(\mathbf{k}, N)$ be the ray class field of $\mathbf{k}$ of modulus N .

Proof of Theorem [1.1](#): We first prove that $\lambda_0 = \lambda(\frac{d+\sqrt{d}}{2})$ is integral. As $j(\tau)$ is an algebraic integer, the equation [\(1.1\)](#) shows that $\lambda(\tau)$ is integral at $\mathfrak{P}$ for any CM point $\tau \in \text{CM}(\mathbf{k}, 2)$ and every prime ideal $\mathfrak{P}$ of $\mathbf{k}(\lambda(\tau)) = H(\mathbf{k}, 2)$ not above 2. So we just need to prove that λ_0 is integral at $\mathfrak{p}$ for every prime $\mathfrak{p}$ of the Hilbert class field $H = \mathbf{k}(j_0)$ of $\mathbf{k}$ above 2 with $j_0 = j(\frac{d+\sqrt{d}}{2})$.

When $d \equiv 5 \pmod{8}$, $[\mathbf{k}(\lambda_0) : \mathbf{k}(j_0)] = 3$ by Lemma [2.1](#). The Galois conjugates of λ_0 over $\mathbf{k}(j_0) = H$ are λ_0 , $\frac{\lambda_0-1}{\lambda_0}$, and $\frac{1}{1-\lambda_0}$. So $N_{\mathbf{k}(\lambda_0)/\mathbf{k}(j_0)}(\lambda_0) = -1$. On the other hand, $2\mathcal{O}_{\mathbf{k}}$ splits completely in $\mathbf{k}(j_0)$ and has then to be ramified in $\mathbf{k}(\lambda_0) = H(\mathbf{k}, 2)$. So for every prime ideal $\mathfrak{p}$ of $\mathbf{k}(j_0)$ above 2, $\mathbf{k}(\lambda_0)_{\mathfrak{p}}/\mathbf{k}(j_0)_{\mathfrak{p}}$ is a ramified local field extension of degree 3. So $N_{\mathbf{k}(\lambda_0)/\mathbf{k}(j_0)}(\lambda_0) = -1$ implies that λ_0 is a unit in $\mathbf{k}(\lambda_0)_{\mathfrak{p}}$. Therefore, λ_0 is integral and actually a unit in $\mathbf{k}(\lambda_0)$. So all six λ -values are units as they are Galois conjugates over $\mathbb{Q}(j_0)$.

When $d \equiv 4 \pmod{8}$, let $\mathfrak{p}_2$ be the prime of $\mathbf{k}$ above 2. Then $\mathfrak{p}_2$ is unramified in H and has to be ramified in $\mathbf{k}(\lambda_0) = H(\mathbf{k}, 2)$ (as $H \neq H(\mathbf{k}, 2)$). In particular, for every prime $\mathfrak{p}$ of H above 2, $\mathfrak{p} = \mathfrak{P}^2$ is ramified in $\mathbf{k}(\lambda_0)$. By Proposition [1.2](#), λ_0 and $1/\lambda_0$ are Galois conjugates over H , and so we have

$$N_{H(\mathbf{k}, 2)_{\mathfrak{P}}/H_{\mathfrak{p}}} \lambda_0 = 1,$$

which implies that λ_0 is a unit in $H(\mathbf{k}, 2)_{\mathfrak{P}}$ for every prime ideal $\mathfrak{P}$ of $H(\mathbf{k}, 2)$. So λ_0 is integral and actually a unit in $H(\mathbf{k}, 2)$.

Next, we assume $d \equiv 0 \pmod{8}$. As the Galois conjugate of λ_0 is $\frac{\lambda_0}{\lambda_0-1}$ by Proposition [1.2](#) the Galois conjugate of $1 - \lambda_0$ is $\frac{1}{1-\lambda_0}$. So the same argument as in the case $d \equiv 4 \pmod{8}$ shows that $1 - \lambda_0$ is integral and a unit in $H(\mathbf{k}, 2)$.

Finally, we assume $d \equiv 1 \pmod{8}$. Let

$$a = \lambda_0(1 - \lambda_0), \quad \omega_0 = -\frac{16}{a}.$$

It is not hard to find that

$$j_0 = \frac{(\omega_0 + 16)^3}{\omega_0}, \quad \text{or } (\omega_0 + 16)^3 - j_0 \omega_0 = 0.$$

Hence, ω_0 is integral by integrality of j_0 . By Proposition [1.2](#), λ_0 and $1 - \lambda_0$ are in the same Galois orbit. By Proposition [1.3](#)

$$|N_{\mathbf{k}(\lambda_0)/\mathbb{Q}} a| = |N_{\mathbf{k}(j_0)/\mathbb{Q}} \lambda_0|^2 = |N(\lambda_0)|^2 = 2^{8h}.$$

Here h is the ideal class number of $\mathbf{k}$. Therefore,

$$|N\omega_0| = \frac{16^{2h}}{|N_{\mathbf{k}(\lambda_0)/\mathbb{Q}}a|} = \frac{16^{2h}}{2^{8h}} = 1.$$

So ω_0 is a unit as claimed. In particular, $a = -\frac{16}{\omega_0}$ is integral, and thus λ_0 is integral with

$$\lambda_0^2 - \lambda_0 - a = 0.$$

This proves the theorem.

We remark that $\omega_2(\tau) = -\frac{16}{\lambda(\tau)(1-\lambda(\tau))}$ is actually a hauptmodul for $X_0(2)$. Its CM values are studied in [YY].

Corollary 2.4. *The following are true.*

- (1) When $d \equiv 5 \pmod{8}$, one has $256|j(\frac{d+\sqrt{d}}{2})|$.
- (2) When $d \equiv 1 \pmod{8}$, $\gcd(j(\frac{d+\sqrt{d}}{2}), 2) = 1$.
- (3) When $d \equiv 0 \pmod{4}$, $2^{6h} \parallel N_{\mathbb{Q}(j_0)/\mathbb{Q}}(j_0)$ for $j_0 = j(\frac{d+\sqrt{d}}{2})$ where h is the class number of $\mathbf{k} = \mathbb{Q}(\sqrt{d})$. Moreover, $\frac{256}{j_0}$ is integral at 2.

Proof. (1) follows from the minimal polynomial equation (1.1) of λ_0 over $\mathbb{Z}(j_0)$ and the integrality of λ_0 . Here we write again $\lambda_0 = \lambda(\frac{d+\sqrt{d}}{2})$.

For (2), recall that $\omega_0 = -\frac{16}{a}$ is a unit with $a = \lambda_0(1 - \lambda_0) \in \mathbb{Q}(j_0)$. So $\text{ord}_{\mathfrak{p}}(a) = 4 \text{ord}_{\mathfrak{p}} 2$ for every prime ideal $\mathfrak{p}$ of $\mathbb{Q}(j_0)$ above 2. This implies

$$\text{ord}_{\mathfrak{p}} j_0 = \text{ord}_{\mathfrak{p}} \frac{256(1-a)^3}{a^2} = 0.$$

So j_0 is a unit at 2. Claim (3) can be proved similarly and we leave it to the reader. $\square$

The explicit calculation suggests that $\frac{j_0}{2^6}$ is integral and is a unit at 2.

3. $\lambda(z_1)$, $1 - \lambda(z_1)$ AND $\lambda(z_1) - \lambda(z_2)$ AS BORCHERDS LIFTINGS

3.1. Brief review of Borchers liftings. Let $L = M_2(\mathbb{Z})$ with the quadratic form $Q(x) = 2 \det x$ and the induced pairing

$$(x, y) = Q(x + y) - Q(x) - Q(y) = 2(y_1x_4 + x_1y_4 - y_2x_3 - x_2y_3).$$

Let $V = M_2(\mathbb{Q})$. Then

$$H = \text{GSpin}(V) = \{g = (g_1, g_2) \in \text{GL}_2 \times \text{GL}_2 : \det g_1 = \det g_2\}$$

acting on V via

$$(g_1, g_2)x = g_1xg_2^{-1}.$$

One has the exact sequence

$$1 \rightarrow \mathbb{G}_m \rightarrow H \rightarrow \text{SO}(V) \rightarrow 1.$$

Let

$$(3.1) \quad \mathcal{L} = \{w \in V_{\mathbb{C}} : (w, w) = 0, \quad (w, \bar{w}) < 0\}.$$

and let $\mathbb{D}$ be the Hermitian symmetric domain of oriented negative 2-planes in $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$. Then one has an isomorphism

$$pr : \mathcal{L}/\mathbb{C}^{\times} \cong \mathbb{D}, \quad w = u + iv \mapsto \mathbb{R}u + \mathbb{R}(-v).$$

For the isotropic matrix $\ell = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \in L$ and $\ell' = \begin{pmatrix} 0 & 0 \\ \frac{1}{N} & 0 \end{pmatrix} \in V$ with $(\ell, \ell') = 1$. We also have the associated tube domain

$$\mathcal{H}_{\ell, \ell'} = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & -z_2 \end{pmatrix} : y_1 y_2 > 0 \right\}, \quad y_i = \text{Im}(z_i),$$

together with

$$w : \mathcal{H}_{\ell, \ell'} \rightarrow \mathcal{L}, \quad w\left(\begin{pmatrix} z_1 & 0 \\ 0 & -z_2 \end{pmatrix}\right) = \begin{pmatrix} z_1 & -2z_1 z_2 \\ \frac{1}{2} & -z_2 \end{pmatrix}.$$

This gives an isomorphism $\mathcal{H}_{\ell, \ell'} \cong \mathcal{L}/\mathbb{C}^{\times}$. We also identify $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ with $\mathcal{H}_{\ell, \ell'}$ by

$$\phi : z = (z_1, z_2) \mapsto \begin{pmatrix} \frac{z_1}{2} & 0 \\ 0 & \frac{z_2}{2} \end{pmatrix}.$$

Note that we use this identification in order to have the following compatibility property and it is also the identification used in the computation of Borcherds products. The following is a special case of [YY Proposition 3.1]

Proposition 3.1. *Define*

$$w_2 : \mathbb{H}^2 \cup (\mathbb{H}^-)^2 \rightarrow \mathcal{L}, \quad w_2(z_1, z_2) = w \circ \phi(z_1, z_2) = \begin{pmatrix} \frac{z_1}{2} & \frac{-z_1 z_2}{2} \\ \frac{1}{2} & \frac{-z_2}{2} \end{pmatrix}.$$

Then the composition $pr \circ w_2$ gives an isomorphism between $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ and $\mathbb{D}$. Moreover, w_2 is $H(\mathbb{R})$ -equivariant, where $H(\mathbb{R})$ acts on $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ via the usual linear fraction:

$$(g_1, g_2)(z_1, z_2) = (g_1(z_1), g_2(z_2)),$$

and acts on $\mathcal{L}$ and $\mathbb{D}$ naturally via its action on V . Moreover, one has

$$(3.2) \quad (g_1, g_2)w_2(z_1, z_2) = \frac{j(g_1, z_1)j(g_2, z_2)}{\nu(g_1, g_2)} w_2(g_1(z_1), g_2(z_2)),$$

where $\nu(g_1, g_2) = \det g_1 = \det g_2$ is the spin character of $H = \text{GSpin}(V)$, and

$$j(g_1, g_2, z_1, z_2) = j(g_1, z_1)j(g_2, z_2) = (c_1 z_1 + d_1)(c_2 z_2 + d_2)$$

is the automorphy factor (of weight $(1, 1)$).

Let $\Gamma = \Gamma(2)$ in this paper, and let

$$K(2) = \{g \in \text{GL}_2(\hat{\mathbb{Z}}) : g \equiv \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}\}$$

and $K = K_{\Gamma(2)} = (K(2) \times K(2)) \cap H(\mathbb{A}_f)$. Let $X_K = H(\mathbb{Q}) \backslash \mathbb{D} \times H(\mathbb{A}_f) / K$ be the associated Shimura variety, then one has

$$X_K \cong Y(2) \times Y(2),$$

where $Y(2) = \Gamma(2) \backslash \mathbb{H}$. We recall that $X(2) = Y(2) \cup \{0, 1, \infty\}$ is the compactified modular curve. Under this identification, the tautological line bundle $\mathcal{L}_K = H(\mathbb{Q}) \backslash \mathcal{L} \times H(\mathbb{A}_f) / K$ over X_K becomes the line bundle of two variable modular forms of weight $(1, 1)$ for $\Gamma(2) \times$

$\Gamma(2)$. Notice that the dual lattice of L is $L' = \frac{1}{2}L$. For each $\mu \in L'/L$ and $m \in Q(\mu) + L$, the associated special divisor $Z(m, \mu)$ is given by

$$Z(m, \mu) = (\Gamma(2) \times \Gamma(2)) \setminus \{(z_1, z_2) : w_2(z_1, z_2) \perp x \text{ for some } x \in \mu + L, Q(x) = m\}$$

Let $S_L = \bigoplus_{\mu \in L'/L} \phi_\mu \subset S(V_{\mathbb{A}_f})$ with $\phi_\mu = \text{Char}(\mu + \hat{L})$, and let ω_L be the Weil representation of $\text{SL}_2(\mathbb{Z})$ on S_L as in [YY]. Let $M_{0, \omega_L}^!$ be the space of S_L -valued weakly holomorphic modular forms of $\text{SL}_2(\mathbb{Z})$ of weight 0 and representation ω_L , i.e., holomorphic functions

$$f : \mathbb{H} \rightarrow S_L$$

such that $f(\gamma\tau) = \omega_L(\gamma)f(\tau)$ and f is meromorphic at the cusp ∞ . It has Fourier expansion

$$(3.3) \quad f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m, \mu) q^m \phi_\mu.$$

The following is a special case of Borcherds' far reaching lifting theorem ([Bor98 Theorem 13.3], see also [YY Theorems 2.1 and 2.2]).

Theorem 3.2. *Let $f(\tau) \in M_{0, \omega_L}^!$ be as above and assume $c(m, \mu) \in \mathbb{Z}$ for $m < 0$. Write $z = (z_1, z_2) \in \mathbb{H}^2$. Then there is a meromorphic modular form $\Psi(z, f)$ of weight (k, k) for $\Gamma(2) \times \Gamma(2)$ with $k = c(0, 0)/2$ (with some characters) such that*

- (1) *one has on the product of open modular curves $Y(2) \times Y(2)$*

$$\text{Div}(\Psi(z, f)^2) = Z(f) = \sum_{m > 0, \mu \in L'/L} c(-m, \mu) Z(m, \mu),$$

here the multiplicity of $Z(m, \mu)$ is 2 if $2\mu = 0$ in L'/L and 1 if $2\mu \neq 0$ in L'/L .

- (2) *Near each cusp $\mathbb{Q}\ell$ of X_K ($\ell \in V$ with $(\ell, \ell) = 0$), $\Psi(z, f)$ has a product expansion of the form:*

$$\Psi(z, f) = Ce((z, \rho(W, f))) \prod_{\substack{\gamma \in M' \\ (\gamma, W) > 0}} \prod_{\substack{\mu \in L'_0/L \\ p(\mu) \in \gamma + M}} [1 - e((\gamma, z) + (\mu, \ell'))]^{c(-Q(\gamma), \mu)}.$$

Here C is a constant with absolute value

$$(3.4) \quad \left| \prod_{\delta \in \mathbb{Z}/N, \delta \neq 0} \left(1 - e\left(\frac{\delta}{N}\right)\right)^{\frac{c(0, \frac{\delta}{N}\ell)}{2}} \right|.$$

Here the unexplained notation are the same as in [YY Section 2], and will be defined in the special case we considered below.

3.2. $\lambda(z_1) - \lambda(z_2)$ as a Borcherds lifting. We choose explicit representatives of cosets $L'/L = \{\mu_j : 0 \leq j \leq 15\}$ with

$$\begin{aligned} \mu_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mu_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mu_2 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \mu_3 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \\ \mu_4 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \mu_5 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \mu_6 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \mu_7 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}\mu_8 &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mu_9 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mu_{10} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \mu_{11} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \\ \mu_{12} &= \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, \mu_{13} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mu_{14} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mu_{15} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.\end{aligned}$$

Take the cusp $\ell_\infty = -e_{12} \in L$ and choose $\ell'_\infty = \frac{1}{2}e_{21} \in L'$ with $(\ell_\infty, \ell'_\infty) = 1$. Here e_{ij} is the 2×2 matrix with (ij) entry 1 and other entries 0. Let $M_\infty = (\mathbb{Q}\ell_\infty + \mathbb{Q}\ell'_\infty)^\perp \cap L = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$. Take $\ell_{M_\infty} = e_{11}$ and $\ell'_{M_\infty} = \frac{1}{2}e_{22} \in M'_\infty$ with $(\ell_{M_\infty}, \ell'_{M_\infty}) = 1$. Furthermore, take $P_\infty = M_\infty \cap (\mathbb{Q}\ell_{M_\infty} + \mathbb{Q}\ell'_{M_\infty})^\perp = 0$ in this case. Finally, take

$$L'_{0\infty} = \{x \in L' : (x, L) \subset 2\mathbb{Z}\},$$

then it is easy to check that

$$(3.5) \quad L'_{0\infty}/L = \{\mu_0, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_{10}, \mu_{15}\}.$$

Let $p_\infty : L'_{0\infty}/L \rightarrow M'_\infty/M_\infty$ be the projection defined in [YY] (2.12)]. Then

$$(3.6) \quad \begin{aligned}p_\infty(\mu_0) &= p_\infty(\mu_6) = 0, & p_\infty(\mu_3) &= p_\infty(\mu_{10}) = \frac{1}{2}e_{22}, \\ p_\infty(\mu_4) &= p_\infty(\mu_7) = \frac{1}{2}e_{11}, & p_\infty(\mu_5) &= p_\infty(\mu_{15}) = \frac{1}{2}I_2.\end{aligned}$$

Here I_2 is the identity matrix of rank 2.

This explains most of the unexplained notations in the last theorem except the Weyl chamber W and Weyl vector $\rho(W, f)$, both of which depend on the choice of f (we refer to [YY] Section 2] for a brief review of them). We first determine the Borcherds products of the constant weakly holomorphic vector valued modular forms which will be used later and are of independent interest.

Proposition 3.3. *Let $M_{0,\omega_L}^{1,0}$ be the subspace of M_{0,ω_L}^1 consisting of constant vectors $f = \sum c(0, \mu_i)\phi_{\mu_i}$. Then*

- (1) *The space $M_{0,\omega_L}^{1,0}$ is of dimension 5 with a basis $\{f_j : 0 \leq j \leq 4\}$, where*

$$\begin{aligned}f_0 &= \phi_{\mu_0} + \phi_{\mu_9} + \phi_{\mu_{10}} + \phi_{\mu_{12}}, \\ f_1 &= \phi_{\mu_1} + \phi_{\mu_7} - \phi_{\mu_{10}} - \phi_{\mu_{12}}, \\ f_2 &= \phi_{\mu_2} - \phi_{\mu_6} - \phi_{\mu_7} + \phi_{\mu_{12}}, \\ f_3 &= \phi_{\mu_3} + \phi_{\mu_6} - \phi_{\mu_9} - \phi_{\mu_{12}}, \\ f_4 &= \phi_{\mu_4} + \phi_{\mu_6} + \phi_{\mu_7} - \phi_{\mu_9} - \phi_{\mu_{10}} - \phi_{\mu_{12}}.\end{aligned}$$

(2) *Their Borchers liftings are given by*

$$\begin{aligned}\Psi(z, 24f_0) &= 2^{12} \Delta(z_2) \Delta(2z_1) (1 - \lambda(z_1))^3 = \Delta(z_2) \omega_2(z_1) \Delta(z_1) (1 - \lambda(z_1))^3, \\ \Psi(z, 24f_1) &= \frac{\omega_2(z_2) (1 - \lambda(z_2))^3}{\omega_2(z_1) (1 - \lambda(z_1))^3} = \lambda(z_1) (1 - \lambda(z_1))^{-2} \lambda(z_2)^{-1} (1 - \lambda(z_2))^2, \\ \Psi(z, 24f_2) &= 2^{12} \frac{1}{\omega_2(z_1) \omega_2(z_2)^2 (1 - \lambda(z_2))^3} = -\lambda(z_1) (1 - \lambda(z_1)) \lambda(z_2)^2 (1 - \lambda(z_2))^{-1}, \\ \Psi(z, 24f_3) &= -2^{-12} \omega_2(z_1)^2 \omega_2(z_2) \lambda(z_1)^3 = \lambda(z_1) (1 - \lambda(z_1))^{-2} \lambda(z_2)^{-1} (1 - \lambda(z_2))^{-1}, \\ \Psi(z, 24f_4) &= -2^{-12} \frac{\omega_2(z_2)^3 \lambda(z_2)^3 (1 - \lambda(z_2))^3}{(1 - \lambda(z_1))^3} = (1 - \lambda(z_1))^{-3}.\end{aligned}$$

Here

$$\omega_2(z) = \frac{-16}{\lambda(z)(1 - \lambda(z))}$$

is a Hauptmodul for $\Gamma_0(2)$. Finally, $\lambda(z_i)$ and $1 - \lambda(z_i)$ can be written as products of integral powers of $\Psi(z, 8f_j)$, $j = 1, 2, 3, 4$, up to a constant multiple.

(3) *If $\Psi(z_1, z_2)$ is a two variable meromorphic modular form for $\Gamma(2) \times \Gamma(2)$ of weight (k, k) with divisor solely supported on the boundary. Then*

$$\Psi(z_1, z_2)^3 = C \Psi(z_1, z_2, f)$$

is a Borchers lifting of some $f \in M_{0, \omega_L}^{1,0}$ up to a constant multiple.

Proof. (sketch) Write $n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $f = \sum c_f(0, \mu_i) \phi_{\mu_i} \in M_{0, \omega_L}^1$ if and only if

$$f(\tau + 1) = \omega_L(n(1))f \quad \text{and} \quad f(-1/\tau) = \omega_L(w)f,$$

i.e.,

$$\sum c_f(0, \mu_i) (e(Q(\mu_i)) - 1) \phi_{\mu_i} = 0$$

and

$$\sum c_f(0, \mu_i) \omega_L(w) \phi_{\mu_i} = \sum c_f(0, \mu_i) \omega_L(w) \phi_{\mu_i}.$$

Now a simple (tedious) linear algebra calculation gives (1). To find the Borchers products of $\Psi(f)$ for $f \in M_{0, \omega_L}^{1,0}$, notice first that since f has no negative terms, there is only one Weyl chamber

$$W = \{\mathbb{R} \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} : a > 0\}.$$

So $\gamma = \begin{pmatrix} -\frac{m}{2} & 0 \\ 0 & \frac{n}{2} \end{pmatrix} \in M'$ satisfies $(\gamma, W) > 0$ if and only if $m, n \geq 0$ are integers not both equal to zero. The Weyl vector is given by, via explicit calculation using the formula [YY (2.17)],

$$\rho(W, f) = \frac{1}{48} \begin{pmatrix} -c_2 & 0 \\ 0 & c_1 \end{pmatrix},$$

with

$$\begin{aligned}c_2 &= 2c_f(0, \mu_0) + 2c_f(0, \mu_6) - c_f(0, \mu_4) - c_f(0, \mu_7), \\ c_1 &= c_f(0, \mu_0) + c_f(0, \mu_6) + c_f(0, \mu_4) + c_f(0, \mu_7).\end{aligned}$$

Finally, the constant $C = C(f)$ can be chosen as $C(f) = 2^{\frac{c_f(0, \mu_6)}{2}}$. So one has by Theorem [3.2](#)

$$(3.7) \quad \Psi(z, f) = 2^{\frac{c_f(0, \mu_6)}{2}} q_1^{\frac{c_1}{48}} q_2^{\frac{c_2}{48}} \cdot \prod_{m=1}^{\infty} (1 - q_2^m)^{c_f(0,0)} (1 + q_2^m)^{c_f(0, \mu_6)} (1 - q_2^{m-\frac{1}{2}})^{c_f(0, \mu_4)} (1 + q_2^{m-\frac{1}{2}})^{c_f(0, \mu_7)} \cdot \prod_{n=1}^{\infty} (1 - q_1^n)^{c_f(0,0)} (1 + q_1^n)^{c_f(0, \mu_6)} (1 - q_1^{n-\frac{1}{2}})^{c_f(0, \mu_3)} (1 + q_1^{n-\frac{1}{2}})^{c_f(0, \mu_{10})}.$$

Now we have the following table

	f_0	f_1	f_2	f_3	f_4
$c_f(0, 0)$	1	0	0	0	0
$c_f(0, \mu_6)$	0	0	-1	1	1
$c_f(0, \mu_4)$	0	0	0	0	1
$c_f(0, \mu_7)$	0	1	-1	0	1
$c_f(0, \mu_3)$	0	0	0	1	0
$c_f(0, \mu_{10})$	1	-1	0	0	-1
c_1	1	1	-2	1	3
c_2	2	-1	-1	2	0
$C(f)$	1	1	$2^{-\frac{1}{2}}$	$2^{\frac{1}{2}}$	$2^{\frac{1}{2}}$

Recall the product formula [\(1.2\)](#), and

$$1 - \lambda(\tau) = \frac{1}{16} q_\tau^{-\frac{1}{2}} \prod_{n \geq 1} \left(\frac{1 + q_\tau^{n-\frac{1}{2}}}{1 + q_\tau^n} \right)^8,$$

we obtain the formulas in (2). We rewrite the formulas as

$$\begin{pmatrix} \log |\Psi(z, 24f_1)| \\ \log |\Psi(z, 24f_2)| \\ \log |\Psi(z, 24f_3)| \\ \log |\Psi(z, 24f_4)| \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 & 2 \\ 1 & 1 & 2 & -1 \\ 1 & -2 & -1 & -1 \\ 0 & -3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \log |\lambda(z_1)| \\ \log |1 - \lambda(z_1)| \\ \log |\lambda(z_2)| \\ \log |1 - \lambda(z_2)| \end{pmatrix}.$$

The matrix is invertible with inverse

$$\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

This proves the last claim of (2): $\lambda(z_1)$ and $1 - \lambda(z_1)$ are also Borcherds products.

To prove (3), notice that $\Psi_1(z) = \Psi(z)/\Psi(z, 2kf_0)$ is a two variable modular function with divisor supported on the boundary of $X(2) \times X(2)$. It is easy to check that such a

function is of the form

$$\Psi_1(z) = C_1 \lambda(z_1)^{m_1} (1 - \lambda(z_2))^{m_2} \lambda(z_2)^{m_3} (1 - \lambda(z_2))^{m_4}$$

with $m_i \in \mathbb{Z}$. Indeed, fix $z_2 \in \mathbb{H}$, then $\Psi_1(z_1, z_2)$ is a meromorphic function on $X(2)$ with poles and zeros only at the cusps $\{0, 1, \infty\}$, so

$$\Psi_1(z_1, z_2) = C(z_2) \lambda(z_1)^{m_1} (1 - \lambda(z_1))^{m_2},$$

where $C(z_2)$ is a meromorphic function of z_2 which can only have zeros or poles at the cusps of $X(2)$. So

$$C(z_2) = C_1 \lambda(z_2)^{m_3} (1 - \lambda(z_2))^{m_4}.$$

So there are some integers b_i by (2) such that

$$\Psi_1(z)^3 = C \Psi(z, f_1)^{b_1} \Psi(z, f_2)^{b_2} \Psi(z, f_3)^{b_3} \Psi(z, f_4)^{b_4}.$$

This proves the proposition. □

Corollary 3.4. *We have the following Borcherds lifting:*

$$\begin{aligned} -\lambda(z_1) &= \Psi(z, 8(f_1 + f_2 + f_3 - f_4)) = \Psi(z, 8(\phi_{\mu_1} + \phi_{\mu_2} + \phi_{\mu_3} - \phi_{\mu_4} - \phi_{\mu_6} - \phi_{\mu_7})), \\ 1 - \lambda(z_1) &= \Psi(z, -8f_4) = \Psi(z, -8(\phi_{\mu_4} + \phi_{\mu_6} + \phi_{\mu_7} - \phi_{\mu_9} - \phi_{\mu_{10}} - \phi_{\mu_{12}})), \\ -\lambda(z_2) &= \Psi(z, 8(f_2 - f_3 + f_4)) = \Psi(z, 8(\phi_{\mu_2} - \phi_{\mu_3} + \phi_{\mu_4} - \phi_{\mu_6} - \phi_{\mu_{10}} + \phi_{\mu_{12}})), \\ 1 - \lambda(z_2) &= \Psi(z, 8(f_1 - f_3)) = \Psi(z, 8(\phi_{\mu_1} - \phi_{\mu_3} - \phi_{\mu_6} + \phi_{\mu_7} + \phi_{\mu_9} - \phi_{\mu_{10}})). \end{aligned}$$

Proposition 3.5. *Let*

$$\begin{aligned} f(\tau) &= -8 \sum_{\gamma \in \Gamma(2) \backslash \mathrm{SL}_2(\mathbb{Z})} \lambda |\gamma \omega_L(\gamma)|^{-1} \phi_{\mu_5} - 4f_0 + 4f_1 + 4f_2 - 4f_3 - 4f_4 \\ &= q^{-\frac{1}{2}} \phi_{\mu_5} + \sum_{m \geq 0, \mu \in L'/L} c_f(m, \mu) q^m \phi_{\mu} \in M_{0, \omega_L}^1 \end{aligned}$$

be a weakly holomorphic vector valued modular form induced from λ with slightly modification as indicated. Then the principal part of f is $q^{-\frac{1}{2}} \phi_{\mu_5}$ and (with a suitable choice of the constant)

$$\Psi(z, f) = \lambda(z_1) - \lambda(z_2),$$

and

$$\begin{aligned} c_f(0, \mu_3) &= c_f(0, \mu_4) = c_f(0, \mu_6) = c_f(0, \mu_7) = c_f(0, \mu_{10}) = -8, \\ c_f(0, \mu_2) &= c_f(0, \mu_9) = 8, \\ c_f(0, \mu_j) &= 0 \text{ for all other } j. \end{aligned}$$

Proof. Let

$$f_L(\tau) = -8 \sum_{\gamma \in \Gamma(2) \backslash \mathrm{SL}_2(\mathbb{Z})} \lambda |\gamma \omega_L(\gamma)|^{-1} \phi_{\mu_5}$$

be the weakly holomorphic vector valued modular form induced from λ . Then one has the following Fourier expansion

$$f_L(\tau) = q^{-\frac{1}{2}}\phi_{\mu_5} + \sum_{j=0}^{15} \sum_{r \in \mathbb{Z}_{\geq 0}} c_L(\frac{r}{2}, \mu_j) q^{\frac{r}{2}} \phi_{\mu_j},$$

with

$$\begin{aligned} c_L(0, \mu_0) &= c_L(0, \mu_2) = c_L(0, \mu_6) = c_L(0, \mu_9) = 4, \\ c_L(0, \mu_1) &= c_L(0, \mu_3) = c_L(0, \mu_4) = c_L(0, \mu_7) = c_L(0, \mu_{10}) = c_L(0, \mu_{12}) = -4. \end{aligned}$$

So $Z(f_L) = Z(\frac{1}{2}, \mu_5) = Y_{\Gamma(2)}^\Delta$ is the diagonal cycle in the open Shimura variety $X_K = Y(2) \times Y(2)$.

We first look at the Borcherds lifting of f_L around the cusp $\mathbb{Q}\ell_\infty$ with $\ell_\infty = -e_{12}$ as at the beginning of this section, and $\ell'_\infty = \frac{1}{2}e_{21}$. The Grassmannian $\text{Gr}(M)$ is cut into two Weyl chambers by $Z(f_L)$, the one whose closure contains $\ell_{M_\infty} = e_{11}$ is

$$W = \{\mathbb{R} \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} : a > 1\}.$$

The associated Weyl vector is $\rho(W, f_L) = 0$. For a vector $\gamma = \begin{pmatrix} -\frac{m}{2} & 0 \\ 0 & \frac{n}{2} \end{pmatrix} \in M'$, $(\gamma, W) > 0$ if and only if

$$m + n \geq 0, \quad n > 0, \quad m \text{ and } n \text{ are not both } 0.$$

Finally, the identification between $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ and $\mathcal{H}_{\ell, \ell'}$ is $\phi(z_1, z_2) = \begin{pmatrix} \frac{z_1}{2} & 0 \\ 0 & -\frac{z_2}{2} \end{pmatrix}$. So Theorem 3.2 gives in this special case

$$(3.8) \quad \Psi(z_1, z_2, f_L) = q_2^{\frac{1}{2}}(1 - q_1^{\frac{1}{2}}q_2^{-\frac{1}{2}}) \prod_{m, n \geq 1} \prod_{\substack{\mu \in L'_{0\infty}/L \\ p_\infty(\mu) = \begin{pmatrix} -\frac{m}{2} & 0 \\ 0 & \frac{n}{2} \end{pmatrix} + M_\infty}} (1 - q_1^{\frac{n}{2}}q_2^{\frac{m}{2}}e((\mu, \ell'_\infty)))^{c(\frac{mn}{2}, \mu)}.$$

Let $\Psi(S(z_1), S(z_2), f_L)$ be the Borcherds product expansion of $\Psi(z, f_L)$ at the cusp $\ell_0 = w\ell_\infty w^{-1} = e_{21}$. So $\ell'_0 = w\ell'_\infty w^{-1} = -\frac{1}{2}e_{12}$. Theorem 3.2 together with the same calculation as above, gives the Borcherds product expansion:

$$(3.9) \quad \Psi(S(z_1), S(z_2), f_L) = q_2^{\frac{1}{2}}(1 - q_1^{\frac{1}{2}}q_2^{-\frac{1}{2}}) \prod_{m, n \geq 1} \prod_{\substack{\mu \in L'_{00}/L \\ p_0(\mu) = \begin{pmatrix} \frac{m}{2} & 0 \\ 0 & -\frac{n}{2} \end{pmatrix} + M_0}} (1 - q_1^{\frac{m}{2}}q_2^{\frac{n}{2}}e((\mu, \ell'_0)))^{c(\frac{mn}{2}, \mu)},$$

where $M_0 = (\mathbb{Q}\ell_0 + \mathbb{Q}\ell'_0)^\perp \cap L = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$, and

$$L'_{00} = \{x \in L' : (x, \ell'_0) \in 2\mathbb{Z}\} = \{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_{12}, \mu_{14}\} + L,$$

and

$$p_0 : L'_{00}/L \rightarrow M'_0/M_0,$$

is the associated projection similar to (3.6).

Now we assume

$$f' = \sum_{i=0}^{15} a_i \phi_{\mu_i}$$

is a constant vector valued modular form such that $\Psi(z_1, z_2, f_L + f') = \lambda(z_1) - \lambda(z_2)$. From the expression (3.8) and (3.9) and comparing the leading terms, we want the Borcherds lifting of f' at the cusp ℓ_∞ to have Weyl vector $\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and the Borcherds lifting of f' at the cusp ℓ_0 to have Weyl vector $\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$. Note that

$$f'_{M_\infty} = (a_0 + a_6)\phi_{\mu_0} + (a_4 + a_7)\phi_{\mu_4},$$

$$f'_{M_0} = (a_0 + a_2)\phi_{\mu_0} + (a_1 + a_3)\phi_{\mu_3},$$

and

$$\ell_{M_\infty} = e_{11}, \ell'_{M_\infty} = \frac{1}{2}e_{22}, \ell_{M_0} = e_{22}, \ell'_{M_0} = \frac{1}{2}e_{11}.$$

Then by the Weyl vector formula in [YY] (2.15)-(2.18)], we have

$$\rho_{\ell'_{M_\infty}} = \frac{a_0 + a_4 + a_6 + a_7}{24} = -1,$$

$$\rho_{\ell_{M_\infty}} = -\frac{a_0 + a_6}{24} + \frac{a_4 + a_7}{48} = \frac{1}{2},$$

$$\rho_{\ell'_{M_0}} = \frac{a_0 + a_1 + a_2 + a_3}{24} = 0,$$

$$\rho_{\ell_{M_0}} = -\frac{a_0 + a_2}{24} + \frac{a_1 + a_3}{48} = 0$$

Since $c(0, f_L) = 4$, we also require $a_0 = -4$ in order to get a modular function. Note that $T \circ f' = f'$ and $S \circ f' = f'$ is the basic requirement for f' to be a vector valued modular form. These linear equations have a unique solution which can be written as

$$f' = -4f_0 + 4f_1 + 4f_2 - 4f_3 - 4f_4.$$

Now it is easy to see that

$$\Psi(z_1, z_2, f_L + f') / (\lambda(z_1) - \lambda(z_2))$$

is holomorphic on $X(2) \times X(2)$ and is thus constant. By comparing the coefficients, we get that $\Psi(z_1, z_2, f_L + f') = \lambda(z_1) - \lambda(z_2)$. $\square$

For the convenience of the reader interested in calculation, we record an explicit expansion for $f(\tau)$ as the following proposition.

Proposition 3.6. *Let*

$$f(\tau) = \sum_{j=0}^{15} f(\tau, \mu_j) e_{\mu_j}$$

be the weakly modular form in Proposition 3.5. Then

$$\begin{aligned} f(\tau, \mu_0) &= 1024q + 24576q^2 + 307200q^3 + 2686976q^4 + 18561024q^5 + \dots \\ f(\tau, \mu_1) &= -1024q - 24576q^2 - 307200q^3 - 2686976q^4 - 18561024q^5 + \dots \\ f(\tau, \mu_2) &= 8 + 1024q + 24576q^2 + 307200q^3 + 2686976q^4 + 18561024q^5 + \dots \\ f(\tau, \mu_3) &= -8 - 1024q - 24576q^2 - 307200q^3 - 2686976q^4 - 18561024q^5 + \dots \\ f(\tau, \mu_4) &= -8 - 1024q - 24576q^2 - 307200q^3 - 2686976q^4 - 18561024q^5 + \dots \\ f(\tau, \mu_5) &= q^{-\frac{1}{2}} + 148q + 5570q^2 + 92120q^3 + 940415q^4 + 7239908q^5 + \dots \\ f(\tau, \mu_6) &= -8 + 1024q + 24576q^2 + 307200q^3 + 2686976q^4 + 18561024q^5 + \dots \\ f(\tau, \mu_7) &= -8 - 1024q - 24576q^2 - 307200q^3 - 2686976q^4 - 18561024q^5 + \dots \\ f(\tau, \mu_8) &= 128q^{\frac{1}{2}} + 5632q^{\frac{3}{2}} + 91904q^{\frac{5}{2}} + 941056q^{\frac{7}{2}} + 7238272q^{\frac{9}{2}} + \dots \\ f(\tau, \mu_9) &= 8 + 1024q + 24576q^2 + 307200q^3 + 2686976q^4 + 18561024q^5 + \dots \\ f(\tau, \mu_{10}) &= -8 - 1024q - 24576q^2 - 307200q^3 - 2686976q^4 - 18561024q^5 + \dots \\ f(\tau, \mu_{11}) &= -128q^{\frac{1}{2}} - 5632q^{\frac{3}{2}} - 91904q^{\frac{5}{2}} - 941056q^{\frac{7}{2}} - 7238272q^{\frac{9}{2}} + \dots \\ f(\tau, \mu_{12}) &= -1024q - 24576q^2 - 307200q^3 - 2686976q^4 - 18561024q^5 + \dots \\ f(\tau, \mu_{13}) &= -128q^{\frac{1}{2}} - 5632q^{\frac{3}{2}} - 91904q^{\frac{5}{2}} - 941056q^{\frac{7}{2}} - 7238272q^{\frac{9}{2}} + \dots \\ f(\tau, \mu_{14}) &= 128q^{\frac{1}{2}} + 5632q^{\frac{3}{2}} + 91904q^{\frac{5}{2}} + 941056q^{\frac{7}{2}} + 7238272q^{\frac{9}{2}} + \dots \\ f(\tau, \mu_{15}) &= 128q^{\frac{1}{2}} + 5632q^{\frac{3}{2}} + 91904q^{\frac{5}{2}} + 941056q^{\frac{7}{2}} + 7238272q^{\frac{9}{2}} + \dots \end{aligned}$$

4. THE BIG CM VALUE FORMULA AND PROOFS OF THE MAIN RESULTS

4.1. Big CM cycles. Let $\mathbf{k}_i = \mathbb{Q}(\sqrt{d_i})$, $i = 1, 2$, be two imaginary quadratic fields of fundamental discriminants d_i , and $E = \mathbf{k}_1 \mathbf{k}_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Let $F = \mathbb{Q}(\sqrt{D})$ be the real quadratic field with $D = d_1 d_2$. We assume in this paper $(d_1, d_2) = 1$. Let $W = E$ with F -quadratic form $Q_F(x) = \frac{2x\bar{x}}{\sqrt{D}}$, and let $W_{\mathbb{Q}} = W$ with $\mathbb{Q}$ -quadratic form $Q(x) = \text{tr}_{F/\mathbb{Q}}(Q_F(x))$. Let $\sigma_1 = 1$ and $\sigma_2 = \sigma$ be two real embeddings of F with $\sigma_i(\sqrt{D}) = (-1)^{i-1} \sqrt{D}$. Then W has signature $(0, 2)$ at σ_2 and $(2, 0)$ at σ_1 respectively, and so $W_{\mathbb{Q}}$ has signature $(2, 2)$. Choose a $\mathbb{Z}$ -basis of $\mathcal{O} = \mathcal{O}_E$ as follows

$$e_1 = 1 \otimes 1, \quad e_2 = \frac{-d_1 + \sqrt{d_1}}{2} = \frac{-d_1 + \sqrt{d_1}}{2} \otimes 1, \quad e_3 = \frac{d_2 + \sqrt{d_2}}{2} = 1 \otimes \frac{d_2 + \sqrt{d_2}}{2}, \quad e_4 = e_2 e_3.$$

We will drop $\otimes$ when there is no confusion. Then it is easy to check that

$$(4.1) \quad (W_{\mathbb{Q}}, Q_{\mathbb{Q}}) \cong (V, Q) = (M_2(\mathbb{Q}), 2 \det), \quad \sum x_i e_i \mapsto \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix}.$$

We will identify $(W_{\mathbb{Q}}, Q_{\mathbb{Q}})$ with the quadratic space $(V, Q) = (M_2(\mathbb{Q}), 2 \det)$. Under this identification, $L = M_2(\mathbb{Z})$ becomes $\mathcal{O}_E$.

Let T be the torus over $\mathbb{Q}$ given by (see [HY12] and also [BKY12] Section 6])

$$T(R) = \{(t_1, t_2) \in (\mathbf{k}_1 \otimes_{\mathbb{Q}} R)^{\times} \times (\mathbf{k}_2 \otimes_{\mathbb{Q}} R)^{\times} : t_1 \bar{t}_1 = t_2 \bar{t}_2\},$$

for any $\mathbb{Q}$ -algebra R . Then there is a map from T to $\mathrm{SO}(W)$ given by $(t_1, t_2) \mapsto t_1/\bar{t}_2$. Define the embedding

$$(4.2) \quad \iota_j : \mathbf{k}_j \rightarrow M_2(\mathbb{Q}), \quad \iota_j(r)(e_{j+1}, e_1)^t = (re_{j+1}, re_1)^t.$$

Then $\iota = (\iota_1, \iota_2)$ gives the embedding from T to H , and one has the commutative diagram:

$$\begin{array}{ccccccc} 1 & \rightarrow & \mathbb{G}_m & \rightarrow & T & \rightarrow & \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(W) \rightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & \mathbb{G}_m & \rightarrow & H & \rightarrow & \mathrm{SO}(V) \rightarrow 1 \end{array}$$

This shows that T is a maximal torus of H . Since W_{σ_2} is of signature $(0, 2)$, it gives two points $z_{\sigma_2}^{\pm}$ in X_K , a big CM point in the sense of [BKY12]. The associated big CM cycles are

$$Z(W, z_{\sigma_2}) = \{z_{\sigma_2}\} \times (T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / K_T)$$

where $z_{\sigma_2} = z_{\sigma_2}^+$,

$$K_T = \{(t_1, t_2) \in T(\mathbb{A}_f) : (\iota_1(t_1), \iota_2(t_2)) \in K(\Gamma(2)) \times K(\Gamma(2))\}.$$

This cycle is defined over $\sigma_2(E) = E$. Let $Z(W)$ be the formal sum of all its Galois conjugates (four of them), which is a big CM cycle of X_K defined over $\mathbb{Q}$. We refer to [BKY12] for details.

Lemma 4.1. *Let $K_j = 1 + 2\hat{\mathcal{O}}_{\mathbf{k}_j} \subset \hat{\mathcal{O}}_{\mathbf{k}_j}^{\times}$, then $\mathrm{Cl}(K_j) = \mathbf{k}_j^{\times} \backslash E_{j,f}^{\times} / K_j$ is the ray class group $\mathrm{Cl}(\mathbf{k}_j, 2)$ of E of modulus $2\mathcal{O}_{\mathbf{k}_j}$. Moreover, one has an isomorphism*

$$p' : T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / K_T \rightarrow \mathrm{Cl}(\mathbf{k}_1, 2) \times \mathrm{Cl}(\mathbf{k}_2, 2), \quad p'([t_1, t_2]) = ([t_1], [t_2]).$$

Proof. First notice that $K_j = \iota_j^{-1}(K(2))$. Let $H(\mathbf{k}_i, N)$ be the ray class field of $\mathbf{k}_i$ of modulus $N\mathcal{O}_{\mathbf{k}_i}$ for $i = 1, 2$ and $N = 1, 2$. By the same argument as that in [YY, Lemma 3.8], it is enough to show $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2) = \mathbb{Q}$. Since $(d_1, d_2) = 1$, we may assume $2 \nmid d_2$. In particular, 2 is unramified in $H(\mathbf{k}_2, 1)$. Every rational prime $p \neq 2$ is unramified in $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2)$ and thus in $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 1)$. Therefore $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 1)$ is unramified over $\mathbb{Q}$ at every finite prime and is thus equal to $\mathbb{Q}$. Now we have

$$[H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2) : \mathbb{Q}] = [H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2) : H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 1)],$$

which divides $[H(\mathbf{k}_2, 2) : H(\mathbf{k}_2, 1)]$. Since $[H(\mathbf{k}_2, 2) : H(\mathbf{k}_2, 1)] = 1$ or 3 by Lemma [2.1], $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2) = \mathbb{Q}$ or $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2)$ is abelian over $\mathbb{Q}$ of degree 3. Since $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2)$ is also unramified over $\mathbb{Q}$ at finite primes away from 2, we have $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2) \subset \mathbb{Q}(\zeta_{2^n})$, which has even degree over $\mathbb{Q}$, a contradiction. Therefore $H(\mathbf{k}_1, 2) \cap H(\mathbf{k}_2, 2) = \mathbb{Q}$. This proves the lemma. $\square$

4.2. Incoherent Eisenstein series and the big CM value formula of Bruinier, Kudla, and Yang. In this subsection, we review the big CM value formula of Bruinier, Kudla, and Yang [BKY12].

Associated to the F -quadratic space W and the additive adelic character $\psi_F = \psi \circ \text{tr}_{F/\mathbb{Q}}$ is a Weil representation $\omega = \omega_{\psi_F}$ of $\text{SL}_2(\mathbb{A}_F)$ on $S(W(\mathbb{A}_F)) = S(V(\mathbb{A}_{\mathbb{Q}}))$. Let $\chi = \chi_{E/F}$ be the quadratic Hecke character of F associated to E/F , then $\chi = \chi_W$ is also the quadratic Hecke character of F associated to W , and there is a $\text{SL}_2(\mathbb{A}_F)$ -equivariant map

$$(4.3) \quad \lambda = \prod \lambda_v : S(W(\mathbb{A}_F)) \rightarrow I(0, \chi), \quad \lambda(\phi)(g) = \omega(g)\phi(0).$$

Here $I(s, \chi) = \text{Ind}_{B_{\mathbb{A}_F}}^{\text{SL}_2(\mathbb{A}_F)} \chi | \cdot |^s$ is the principal series, whose sections (elements) are smooth functions Φ on $\text{SL}_2(\mathbb{A}_F)$ satisfying the condition

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1}\Phi(g, s), \quad b \in \mathbb{A}_F, \text{ and } a \in \mathbb{A}_F^\times.$$

Here $B = NM$ is the standard Borel subgroup of SL_2 . Such a section is called factorizable if $\Phi = \otimes \Phi_v$ with $\Phi_v \in I(s, \chi_v)$. It is called standard if $\Phi|_{\text{SL}_2(\mathcal{O}_F)\text{SO}_2(\mathbb{R})^2}$ is independent of s . For a standard section $\Phi \in I(s, \chi)$, its associated Eisenstein series is defined for $\Re(s) \gg 0$ as

$$E(g, s, \Phi) = \sum_{\gamma \in B(F) \backslash \text{SL}_2(F)} \Phi(\gamma g, s).$$

For $\phi \in S(V_f) = S(W_f)$, let Φ_f be the standard section associated to $\lambda_f(\phi) \in I(0, \chi_f)$, here the subscript f means the finite part of the adèle. For each real embedding $\sigma_i : F \hookrightarrow \mathbb{R}$, let $\Phi_{\sigma_i} \in I(s, \chi_{\mathbb{C}/\mathbb{R}}) = I(s, \chi_{E_{\sigma_i}/F_{\sigma_i}})$ be the unique ‘weight one’ eigenvector of $\text{SL}_2(\mathbb{R})$ given by

$$\Phi_{\sigma_i}(n(b)m(a)k_\theta) = \chi_{\mathbb{C}/\mathbb{R}}(a)|a|^{s+1}e^{i\theta},$$

for $b \in \mathbb{R}$, $a \in \mathbb{R}^\times$, and $k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \text{SO}_2(\mathbb{R})$. We define for $\vec{\tau} = (\tau_1, \tau_2) \in \mathbb{H}^2$

$$E(\vec{\tau}, s, \phi) = N(\vec{v})^{-\frac{1}{2}} E(g_{\vec{\tau}}, s, \Phi_f \otimes (\otimes_{1 \leq i \leq 2} \Phi_{\sigma_i})),$$

where $\vec{v} = \text{Im}(\vec{\tau})$, $N(\vec{v}) = \prod_i v_i$, and $g_{\vec{\tau}} = (n(u_i)m(\sqrt{v_i}))_{1 \leq i \leq 2}$. It is a (non-holomorphic) Hilbert modular form of scalar weight 1 for some congruence subgroup of $\text{SL}_2(\mathcal{O}_F)$. Following [BKY12], we further normalize

$$E^*(\vec{\tau}, s, \phi) = \Lambda(s+1, \chi) E(\vec{\tau}, s, \phi),$$

where ∂_F is the different of F , $d_{E/F}$ is the relative discriminant of E/F , and

$$(4.4) \quad \Lambda(s, \chi) = A^{\frac{s}{2}} (\pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2}))^2 L(s, \chi) = \Lambda(s, \chi_{\mathbf{k}_1/\mathbb{Q}}) \Lambda(s, \chi_{\mathbf{k}_2/\mathbb{Q}}), \quad A = N_{F/\mathbb{Q}}(\partial_F d_{E/F}) = D.$$

$\Lambda(s, \chi)$ is a holomorphic function of s with functional equation

$$\Lambda(s, \chi) = \Lambda(1-s, \chi),$$

and

$$\Lambda(1, \chi) = \Lambda(0, \chi) = L(0, \chi) = 2^{2-\delta} \frac{h(E)}{w(E)h(F)} = 4 \frac{h(\mathbf{k}_1)h(\mathbf{k}_2)}{w(\mathbf{k}_1)w(\mathbf{k}_2)} \in \mathbb{Q}^\times,$$

where $2^\delta = [\mathcal{O}_E^\times : \mu(E)\mathcal{O}_F^\times] = 1, 2$.

The Eisenstein series is incoherent in the sense of Kudla ([Kud97]). This forces $E^*(\vec{\tau}, 0, \phi) = 0$ automatically.

Proposition 4.2. ([BKY12 Proposition 4.6]) *Let $\phi \in S(V_f) = S(W_f)$. For a totally positive element $t \in F_+^\times$, let $a(t, \phi)$ be the t -th Fourier coefficient of $E^{*,\prime}(\vec{\tau}, 0, \phi)$ and write the constant term of $E^{*,\prime}(\vec{\tau}, 0, \phi)$ as*

$$\phi(0)\Lambda(0, \chi) \log N(\vec{v}) + a_0(\phi).$$

Let

$$\mathcal{E}(\tau, \phi) = a_0(\phi) + \sum_{n \in \mathbb{Q}_{>0}} a_n(\phi) q^n,$$

where (for $n > 0$)

$$a_n(\phi) = \sum_{t \in F_+^\times, \text{tr}_{F/\mathbb{Q}} t = n} a(t, \phi).$$

Here $F_+^\times$ consists of all totally positive elements in F . Then, writing $\tau^\Delta = (\tau, \tau) \in \mathbb{H}^2$,

$$E^{*,\prime}(\tau^\Delta, 0, \phi) - \mathcal{E}(\tau, \phi) - 2\phi(0)\Lambda(0, \chi) \log v$$

is of exponential decay as v goes to infinity. Moreover, for $n > 0$

$$a_n(\phi) = \sum_p a_{n,p}(\phi) \log p$$

with $a_{n,p}(\phi) \in \mathbb{Q}(\phi)$, the subfield of $\mathbb{C}$ generated by the values $\phi(x)$, $x \in V(\mathbb{A}_f)$.

Remark 4.3. There is a minor mistake in [BKY12 Proposition 4.6] about the constant. The corrected form is

$$E_0^{*,\prime}(\vec{\tau}, 0, \phi) = \phi(0)\Lambda(0, \chi) \log N(\vec{v}) + a_0(\phi)$$

(i.e., $a_0(\phi)$ might not be a multiple of $\phi(0)$). Indeed, direct calculation gives

$$E_0^*(\vec{\tau}, s, \phi) = \phi(0)\Lambda(s+1, \chi)(N(\vec{v}))^{\frac{s}{2}} - (N(\vec{v}))^{-\frac{s}{2}}\Lambda(s, \chi)\tilde{W}_{0,f}(s, \phi),$$

where (when ϕ is factorizable)

$$\tilde{W}_{0,f}(s, \phi) = \prod_{\mathfrak{p} \nmid \infty} \tilde{W}_{0,\mathfrak{p}}(s, \phi_{\mathfrak{p}}) = \prod_{\mathfrak{p} \nmid \infty} \frac{|A|_{\mathfrak{p}}^{-\frac{1}{2}} L_{\mathfrak{p}}(s+1, \chi)}{\gamma(W_{\mathfrak{p}}) L_{\mathfrak{p}}(s, \chi)} W_{0,\mathfrak{p}}(s, \phi_{\mathfrak{p}})$$

is the product of re-normalized local Whittaker functions. With this notation, one has

$$(4.5) \quad a_0(\phi) = -\Lambda(0, \chi)\tilde{W}'_{0,f}(0, \phi) - 2\phi(0)\Lambda'(0, \chi).$$

Here we used the fact that $\tilde{W}_{0,f}(0, \phi) = \phi(0)$, which follows from $E_0^*(\tau, 0, \phi) = 0$.

Notice that $a(t, \phi_{\mu}) = 0$ automatically unless $\mu + \hat{L}$ represents t , i.e., $t - Q_F(\mu) \in \partial_F^{-1} \mathcal{O}_F$. The following is now a special case of the big CM value formula of Bruinier, Kudla, and Yang ([BKY12 Theorem 5.2]).

Theorem 4.4. Assume $(d_1, d_2) = 1$. Let $\tau_i = \frac{d_i + \sqrt{d_i}}{2}$, $i = 1, 2$, and let

$$f(\tau) = \sum_{\mu \in L'/L} f_\mu(\tau) \phi_\mu = \sum c(m, \mu) q^m \phi_\mu \in M_{0, \omega_L}^1$$

be a weakly holomorphic modular form of $\mathrm{SL}_2(\mathbb{Z})$ of weight 0 with respect to the Weil representation ω_L and $c(0, 0) = 0$. Let $\Psi(z, f)$ be its Borcherds lifting. Then

$$\begin{aligned} & - \sum_{[\mathbf{a}_i] \in \mathrm{Cl}(\mathbf{k}_i, 2)} \left(\log |\Psi(\tau_1^{\sigma_{\mathbf{a}_1}}, \tau_2^{\sigma_{\mathbf{a}_2}}, f)|^4 + \log |\Psi((- \bar{\tau}_1)^{\sigma_{\mathbf{a}_1}}, \tau_2^{\sigma_{\mathbf{a}_2}}, f)|^4 \right. \\ & \quad \left. + \log |\Psi(\tau_1^{\sigma_{\mathbf{a}_1}}, (- \bar{\tau}_2)^{\sigma_{\mathbf{a}_2}}, f)|^4 + \log |\Psi((- \bar{\tau}_1)^{\sigma_{\mathbf{a}_1}}, (- \bar{\tau}_2)^{\sigma_{\mathbf{a}_2}}, f)|^4 \right) \\ & = 2r(d_1)r(d_2) \left(\sum_{\substack{0 \neq \mu \in L'/L \\ m \geq 0 \\ m \equiv Q(\mu) \pmod{1}}} c(-m, \mu) a_m(\phi_\mu) \right). \end{aligned}$$

Proof. One has by [BKY12, Theorem 5.2]

$$-\log |\Psi(Z(W), f)|^4 = C(W, K) \left(\sum_{\substack{0 \neq \mu \in L'/L, m \geq 0 \\ m \equiv Q(\mu) \pmod{1}}} c(-m, \mu) a_m(\phi_\mu) \right).$$

Here

$$C(W, K) = \frac{\deg(Z(W, z_{\sigma_2}^\pm))}{\Lambda(0, \chi)}.$$

On the other hand, one has $z_{\sigma_2} = (\tau_1, \tau_2)$ and

$$[z_{\sigma_2}, t_1^{-1}, t_2^{-1}] = (\tau_1^{\sigma_{t_1}}, \tau_2^{\sigma_{t_2}})$$

by [YY, Lemma 3.4, Proposition 3.6]. So $-\log |\Psi(Z(W), f)|^4$ is equal to the left hand side of the identity in the theorem by Lemma 4.1. On the other hand, Lemma 4.1 also implies

$$\begin{aligned} C(W, K) &= \frac{2h(\mathbf{k}_1, 2)h(\mathbf{k}_2, 2)}{\Lambda(0, \chi_{\mathbf{k}_1/\mathbb{Q}})\Lambda(0, \chi_{\mathbf{k}_2/\mathbb{Q}})} \\ &= 2 \prod_{i=1}^2 \frac{2h(\mathbf{k}_i, 2)}{w_i h(\mathbf{k}_i)} = 2r(d_1)r(d_2). \end{aligned}$$

This proves the theorem. □

4.3. The norm of $\lambda(\tau_1) - \lambda(\tau_2)$. Combining Theorem 4.4 and Lemma 2.1, we have

Proposition 4.5. Assume $(d_1, d_2) = 1$. Let $\tau_j = \frac{d_j + \sqrt{d_j}}{2}$ with $j = 1, 2$, and $\delta(D) = 1$ or 2 depending on whether D is odd or even. Then

$$\log |N(\lambda(\tau_1) - \lambda(\tau_2))| = -\frac{r(d_1)r(d_2)}{2\delta(D)} \sum_{\substack{t = \frac{m + \sqrt{D}}{4} \in \frac{1}{2}\mathcal{O}_F \\ |m| < \sqrt{D} \\ t - 2\mu_5\bar{\mu}_5 \in \mathcal{O}_F}} a\left(\frac{t}{\sqrt{D}}, \phi_{\mu_5}\right) - \frac{r(d_1)r(d_2)}{2\delta(D)} a_0,$$

where

$$a_0 = 8a_0(\phi_{\mu_2}) - 8a_0(\phi_{\mu_3}) - 8a_0(\phi_{\mu_4}) - 8a_0(\phi_{\mu_6}) - 8a_0(\phi_{\mu_7}) + 8a_0(\phi_{\mu_9}) - 8a_0(\phi_{\mu_{10}}).$$

Remark 4.6. The appearance of $\delta(D)$ is due to the fact that there is one Galois orbit in $Z(W)$ when D is odd and two when D is even.

4.4. Computing $a(\frac{t}{\sqrt{D}}, \phi_{\mu_5})$. To prove the main theorem of this paper, we need to compute $a(\frac{t}{\sqrt{D}}, \phi_{\mu_5})$ for $t = \frac{m + \sqrt{D}}{4} \in \frac{1}{2}\mathcal{O}_F$ with $|m| < \sqrt{D}$, which occupies this subsection, and $a_0(\phi_\mu)$ for various $\mu = \mu_j$, which will occupy the next subsection. We first set up some notation. Let $\gamma(W_v) = \gamma(W_v, \phi_v)$ be the local Weil index, which is an 8-th root of unity. Then it is known (see for example [Kud94] (3.4))

$$\gamma(W_{\sigma_2}) = i = \gamma(W_{\sigma_1})^{-1}, \quad \prod_{\mathfrak{p} \nmid \infty} \gamma(W_{\mathfrak{p}}) = 1.$$

We normalize

$$W_{t,\mathfrak{p}}^*(s, \phi) = |D|_{\mathfrak{p}}^{-\frac{s+1}{4}} L_{\mathfrak{p}}(s+1, \chi_{\mathfrak{p}}) \frac{W_{t,\mathfrak{p}}(s, \phi)}{\gamma(W_{\mathfrak{p}})}$$

with

$$W_{t,\mathfrak{p}}(s, \phi) = \int_{F_{\mathfrak{p}}} \omega(w_n(b)) \phi_{\mathfrak{p}}(0) |a(w_n(b))|_{\mathfrak{p}}^s \psi_{\mathfrak{p}}(-tb) db,$$

and similarly

$$W_{t,\sigma_j}^*(\tau_j, s, \Phi_{\sigma_j}) = v_j^{-1/2} \pi^{-\frac{s+2}{2}} \Gamma\left(\frac{s+2}{2}\right) \int_{\mathbb{R}} \Phi_{\sigma_j}(w_n(b) g_{\tau_j}, s) \psi(-bt) db.$$

Then we have

$$(4.6) \quad E_t^*(\vec{\tau}, s, \phi) = \prod_{j=1}^2 W_{t,\sigma_j}^*(\tau_j, s, \Phi_{\sigma_j}) \prod_{\mathfrak{p} \nmid \infty} W_{t,\mathfrak{p}}^*(s, \phi).$$

Recall also ([KRY99] Proposition 2.6) that

$$W_{\frac{t}{\sqrt{D}}, \sigma_j}^*(\tau_j, 0, \Phi_{\sigma_j}) = 2i.$$

Let $\text{Diff}(W, t/\sqrt{D})$ be the set of prime ideals $\mathfrak{p}$ of F such that $W_{\mathfrak{p}}$ does not represent $t/\sqrt{D}$. Then $\mathfrak{p} \in \text{Diff}(W, t/\sqrt{D})$ if and only if $t \neq 2u\bar{u}$ for any $u \in E_{\mathfrak{p}}^\times$, i.e., $\mathfrak{p}$ is inert in E/F and $\text{ord}_{\mathfrak{p}}(2t)$ is odd (recall that E/F is unramified), i.e.

$$\text{Diff}(W, t/\sqrt{D}) = \{\mathfrak{p} \text{ inert in } E/F : \text{ord}_{\mathfrak{p}}(2t) \equiv 1 \pmod{2}\}.$$

It is a finite set of odd order. It is well-known that whenever $\mathfrak{p} \in \text{Diff}(W, t/\sqrt{D})$, one has $W_{t,\mathfrak{p}}(0, \phi) = 0$ for all $\phi \in S(W_{\mathfrak{p}})$ (see for example [YY] Lemma 2.7). So $a(t/\sqrt{D}, \phi) = 0$ unless $\text{Diff}(W, t/\sqrt{D}) = \{\mathfrak{p}\}$ has exactly one prime. In this case, we have

$$(4.7) \quad a\left(\frac{t}{\sqrt{D}}, \phi_{\mu_5}\right) = -4W_{\frac{t}{\sqrt{D}},\mathfrak{p}}^{*,\psi'}(0, \phi_{\mu_5}) \prod_{q \nmid \mathfrak{p}\infty} W_{\frac{t}{\sqrt{D}},q}^*(0, \phi_{\mu_5}).$$

Let $\psi'(x) = \psi_F(\frac{x}{\sqrt{D}})$, which is unramified at every prime ideal of F , and $W' = W$ with F -quadratic form $Q'_F(x) = 2x\bar{x}$. Then the Weil representations associated to (W, Q_F, ψ_F) and (W', Q', ψ') are the same, and

$$W_{\frac{t}{\sqrt{D}},\mathfrak{p}}^{\psi_F}(s, \phi) = |D|_{\mathfrak{p}}^{\frac{1}{4}} W_{t,\mathfrak{p}}^{\psi'}(s, \phi)$$

by (5.9). So we have in our case

$$(4.8) \quad W_{\frac{t}{\sqrt{D}},\mathfrak{p}}^*(s, \phi) = W_{t,\mathfrak{p}}^{*,\psi'}(s, \phi) := |D|_{\mathfrak{p}}^{-\frac{s}{4}} L(s+1, \chi_{\mathfrak{p}}) \frac{W_{t,\mathfrak{p}}^{\psi'}(s, \phi)}{\gamma(W'_{\mathfrak{p}})}.$$

Notice that (ϕ_{μ_5}, ψ') is unramified at every prime $\mathfrak{p} \nmid 2$, and we have by Proposition 5.7

$$W_{\frac{t}{\sqrt{D}},\mathfrak{p}}^*(0, \phi_{\mu_5}) = \rho_{\mathfrak{p}}(t) = \begin{cases} 1 + \text{ord}_{\mathfrak{p}}(t) & \text{if } E/F \text{ split at } \mathfrak{p}, \\ \frac{1+(-1)^{\text{ord}_{\mathfrak{p}}(t)}}{2} & \text{if } E/F \text{ inert at } \mathfrak{p}. \end{cases}$$

In particular, $W_{\frac{t}{\sqrt{D}},\mathfrak{p}}^*(0, \phi_{\mu_5}) = 0$ if and only if $\mathfrak{p} \in \text{Diff}(W', t) = \text{Diff}(W, t/\sqrt{D})$. In such a case, one has by the same proposition

$$W_{\frac{t}{\sqrt{D}},\mathfrak{p}}^{*,\psi'}(0, \phi_{\mu_5}) = \frac{1 + \text{ord}_{\mathfrak{p}}(t)}{2}.$$

So we have proved the following formula: when $\text{Diff}(W, t/\sqrt{D}) = \{\mathfrak{p}\}$, we have

$$(4.9) \quad a\left(\frac{t}{\sqrt{D}}, \phi_{\mu_5}\right) = -4 \begin{cases} \frac{1+\text{ord}_{\mathfrak{p}}(t\mathcal{O}_F)}{2} \log N(\mathfrak{p}) \prod_{q \nmid 2\mathfrak{p}} \rho_q(t) \prod_{q \nmid 2} W_{t,q}^{*,\psi'}(0, \phi_{\mu_5}) & \text{if } \mathfrak{p} \nmid 2, \\ W_{t,\mathfrak{p}}^{*,\psi'}(0, \phi_{\mu_5}) \prod_{q \nmid 2} \rho_q(t) \prod_{q \nmid 2, q \neq \mathfrak{p}} W_{t,q}^{*,\psi'}(0, \phi_{\mu_5}) & \text{if } \mathfrak{p} \mid 2. \end{cases}$$

Now we focus on the local calculation at $\mathfrak{p} \mid 2$.

Lemma 4.7. *Let $t = \frac{m+\sqrt{D}}{4} \in \frac{1}{2}\mathcal{O}_F - \mathcal{O}_F$ be as above, and let $\mathfrak{p}$ be a prime of F above 2. Then $W_{t,\mathfrak{p}}^{*,\psi'}(s, \phi_{\mu_5}) = 0$ unless $\alpha(\mu_5, t) = 2\mu_5\bar{\mu}_5 - t \in \mathcal{O}_{\mathfrak{p}}$. Assuming $\alpha(\mu_5, t) \in \mathcal{O}_{\mathfrak{p}}$, we have the following.*

- (1) *When F/Q is inert at 2 with prime $\mathfrak{p}$ over 2. Then $\mathfrak{p} \notin \text{Diff}(W', t)$, $\text{ord}_{\mathfrak{p}}(2t) = 0$, and*

$$W_{t,\mathfrak{p}}^{*,\psi'}(0, \phi_{\mu_5}) = \frac{1}{4} L_{\mathfrak{p}}(1, \chi_{E/F}) = \frac{1}{4} L_{\mathfrak{p}}(1, \chi_{E/F}) \rho_{\mathfrak{p}}(2t).$$

- (2) *When F/Q is ramified at 2 with $2\mathcal{O}_F = \mathfrak{p}^2$. Then $\mathfrak{p} \notin \text{Diff}(W', t)$, and*

$$W_{t,\mathfrak{p}}^{*,\psi'}(0, \phi_{\mu_5}) = \frac{1}{4} L_{\mathfrak{p}}(1, \chi_{E/F}).$$

Moreover, assuming $d_1 \equiv 0 \pmod{4}$, one has

$$\rho_{\mathfrak{p}}(2t) = \begin{cases} 2 & \text{if } d_2 \equiv 1 \pmod{8}, \\ 1 & \text{if } d_2 \equiv 5 \pmod{8}. \end{cases}$$

- (3) When $F/\mathbb{Q}$ is split at 2, let δ be the square root of D in $\mathbb{Z}_2$ with $\delta \equiv 1 \pmod{4}$, and let $2\mathcal{O}_F = \mathfrak{p}_t \tilde{\mathfrak{p}}_t$ with $2t \in \mathfrak{p}_t$ and $2t \notin \tilde{\mathfrak{p}}_t$. Then
- (a) one has $\tilde{\mathfrak{p}}_t \notin \text{Diff}(W', t)$, $\text{ord}_{\tilde{\mathfrak{p}}_t}(2t) = 0$, and

$$W_{t, \tilde{\mathfrak{p}}_t}^{*, \psi'}(0, \phi_{\mu_5}) = \frac{1}{2} L_{\tilde{\mathfrak{p}}_t}(1, \chi_{E/F}) = \frac{1}{2} L_{\tilde{\mathfrak{p}}_t}(1, \chi_{E/F}) \rho_{\tilde{\mathfrak{p}}_t}(2t).$$

(b)

$$W_{t, \mathfrak{p}_t}^{*, \psi'}(0, \phi_{\mu_5}) = \rho_{\mathfrak{p}_t}(t/2) = \rho_{\mathfrak{p}_t}(2t \mathfrak{p}_t^{-2}).$$

In particular, the value is zero if and only if $\text{ord}_{\mathfrak{p}_t}(2t)$ is odd and E/F is inert at $\mathfrak{p}_t$ (i.e. $\mathfrak{p}_t \in \text{Diff}(W', t)$). In this case, one has

$$W_{t, \mathfrak{p}_t}^{*, \psi', '}(0, \phi_{\mu_5}) = \frac{1}{2} (1 + \text{ord}_{\mathfrak{p}_t}(t/2) + L_{\mathfrak{p}_t}(1, \chi_{E/F})) \log N(\mathfrak{p}_t).$$

Proof. We use the notation in Section 5 and shorten $\mu = \mu_5 = \frac{-d_1 + d_2 + \sqrt{d_1} + \sqrt{d_2}}{4}$. In particular,

$$\alpha(\mu, t) = -\frac{m + D}{4} + \frac{d_1^2 - d_1 + d_2^2 - d_2}{8} \in \mathcal{O}_{\mathfrak{p}},$$

and so $a(\mu, t) = \text{ord}_{\mathfrak{p}}(\alpha(\mu, t)) \geq 0$.

Case 1: We first assume that $F/\mathbb{Q}$ is non-split at 2 with $\mathfrak{p}$ being the unique prime ideal of F over 2. Since $t \notin \mathcal{O}_{\mathfrak{p}}$ and $\alpha(\mu, t) \in \mathcal{O}_{\mathfrak{p}}$, we have $2\mu\bar{\mu} \notin \mathcal{O}_{\mathfrak{p}}$ and thus $\text{ord}_{E_{\mathfrak{p}}}(2\mu) = 0$, where for $z \in E_{\mathfrak{p}}$

$$\text{ord}_{E_{\mathfrak{p}}}(z) = \begin{cases} \text{ord}_{\mathfrak{p}}(z) & \text{if } \mathfrak{p} \text{ is inert in } E, \\ \min(\text{ord}_{\mathfrak{B}}(z), \text{ord}_{\tilde{\mathfrak{B}}}(z)) & \text{if } \mathfrak{p}\mathcal{O}_E = \mathfrak{B}\tilde{\mathfrak{B}} \text{ is split in } E. \end{cases}$$

So $a(\mu, t) \geq \text{ord}_{E_{\mathfrak{p}}}(2\mu)$ and Proposition 5.3 implies

$$\frac{W_{t, \mathfrak{p}}^{\psi'}(0, \phi_{\mu})}{\gamma(W', \psi') \text{vol}(\mathcal{O}_{E_{\mathfrak{p}}}, d_{W'}x)} = 1.$$

Here $d_{W'}x$ is the self-dual Haar measure on $E_{\mathfrak{p}} = W'$ with respect to $(x, y) \mapsto \psi'(2x\bar{y})$, and so the volume

$$\text{vol}(\mathcal{O}_{E_{\mathfrak{p}}}, d_{W'}x) = 1/4.$$

Therefore,

$$W_{t, \mathfrak{p}}^{*, \psi'}(0, \psi_{\mu}) = \frac{1}{4} L_{\mathfrak{p}}(1, \chi_{E/F}).$$

as claimed. Now we check

$$\rho_{\mathfrak{p}}(2t) = \begin{cases} 2 & \text{if } 4|d_i, d_j \equiv 1 \pmod{8}, i + j = 3. \\ 1 & \text{otherwise.} \end{cases}$$

Assuming $4|d_1$ (the case $4|d_2$ is the same) first. We have (notice that $D \equiv d_1 \pmod{8}$)

$$N_{E/F}(2\mu) = \frac{-D + \sqrt{D}}{2} + \frac{d_1^2 - d_1}{4} + \frac{d_2^2 - d_2}{4},$$

and so

$$\text{ord}_{\mathfrak{p}}(N_{E/F}(2\mu)) = \begin{cases} 1 & \text{if } d_2 \equiv 1 \pmod{8}, \\ 0 & \text{if } d_2 \equiv 5 \pmod{8}. \end{cases}$$

Since

$$2t - N_{E/F}(2\mu) = -2\alpha(\mu, t) \in 2\mathcal{O}_F = \mathfrak{p}^2,$$

we have that $\text{ord}_{\mathfrak{p}}(2t)$ is 1 or 0 depending whether $d_2 \equiv 1 \pmod{8}$ or $5 \pmod{8}$. This proves the ramified case.

Next assume that $F/\mathbb{Q}$ is inert at 2. In this case, we have (notice $\mathfrak{p} = 2\mathcal{O}_F$)

$$\text{ord}_{\mathfrak{p}}(N_{E/F}(2\mu)) = \text{ord}_{\mathfrak{p}}(1 + \frac{-D + \sqrt{D}}{2}) = 0,$$

and thus $\text{ord}_{\mathfrak{p}}(2t) = 0$, $\rho_{\mathfrak{p}}(2t) = 1$. So we have proved (1) and (2) of the lemma.

Case 2: We now assume that $F/\mathbb{Q}$ is split at 2, i.e. $D \equiv 1 \pmod{8}$. This implies $d_1 \equiv d_2 \equiv 1, 5 \pmod{8}$. Let $\delta \equiv 1 \pmod{4}$ be one square root of D in $\mathbb{Z}_2$. Let $\mathfrak{p}_t$ be the prime of F above 2 such that $(\sqrt{D})_{\mathfrak{p}_t} = \delta$, so $(\sqrt{D})_{\bar{\mathfrak{p}}_t} = -\delta$. It is easy to check (recall $\sqrt{d_1}\sqrt{d_2} = -\sqrt{D}$)

$$\begin{aligned} \mu &= (\mu_{\mathfrak{p}_t}, \mu_{\bar{\mathfrak{p}}_t}) \in \mathcal{O}_{E_{\mathfrak{p}_t}} \times (\frac{1}{2}\mathcal{O}_{E_{\bar{\mathfrak{p}}_t}} - \mathcal{O}_{E_{\bar{\mathfrak{p}}_t}}), \\ 2\mu\bar{\mu} &\in \mathcal{O}_{\mathfrak{p}_t} \times (\frac{1}{2}\mathcal{O}_{\bar{\mathfrak{p}}_t} - \mathcal{O}_{\bar{\mathfrak{p}}_t}), \\ t &\in \mathcal{O}_{\mathfrak{p}_t} \times (\frac{1}{2}\mathcal{O}_{\bar{\mathfrak{p}}_t} - \mathcal{O}_{\bar{\mathfrak{p}}_t}), \end{aligned}$$

as $\alpha(\mu, t) \in \mathcal{O}_F$. So Proposition 5.3 asserts ($\beta = 2$ and $\text{vol}(\mathcal{O}_{E_{\mathfrak{p}_t}}, d_{\beta}x) = \text{vol}(\mathcal{O}_{E_{\bar{\mathfrak{p}}_t}}, d_{\beta}x) = 1/2$ in this case)

$$2 \frac{W_{t, \bar{\mathfrak{p}}_t}^{\psi'}(0, \phi_{\mu})}{\gamma(W')} = 1,$$

i.e.,

$$W_{t, \bar{\mathfrak{p}}_t}^{*, \psi'}(0, \phi_{\mu}) = \frac{1}{2}L_{\bar{\mathfrak{p}}_t}(1, \chi_{E/F}) = \frac{1}{2}L_{\bar{\mathfrak{p}}_t}(1, \chi_{E/F})\rho_{\bar{\mathfrak{p}}_t}(2t),$$

as $\text{ord}_{\bar{\mathfrak{p}}_t}(2t) = 0$.

On the other hand, Proposition 5.7(2) gives when $t \in \mathfrak{p}_t$

$$2L_{\mathfrak{p}_t}(1, \chi_{E/F}) \frac{W_{t, \mathfrak{p}_t}^{\psi'}(s, \phi_{\mu})}{\gamma(W')} = L_{\mathfrak{p}_t}(1, \chi_{E/F})(1 - 2^{-s}) + 2 \sum_{0 \leq n \leq \text{ord}_{\mathfrak{p}_t}(t/2)} (\chi_{\mathfrak{p}_t}(\pi_t)2^{-s})^n.$$

Here π_t is a uniformizer of $F_{\mathfrak{p}_t}$. In particular,

$$W_{t, \mathfrak{p}_t}^{*, \psi'}(0, \phi_{\mu}) = \rho_{\mathfrak{p}_t}(t/2) = \rho_{\mathfrak{p}_t}(2t\mathfrak{p}_t^{-2}).$$

Moreover, the value is zero if and only if $\mathfrak{p}_t$ is inert in E/F (i.e., $d_i \equiv 5 \pmod{8}$), and $\text{ord}_{\mathfrak{p}_t}(2t)$ is odd, i.e., $\mathfrak{p}_t \in \text{Diff}(W', t)$. In such a case,

$$W_{t, \mathfrak{p}_t}^{*, \psi', \prime}(0, \phi_\mu) = \frac{1}{2} \left(1 + \text{ord}_{\mathfrak{p}_t}(t/2) + L_{\mathfrak{p}_t}(1, \chi_{E/F}) \right) \log N(\mathfrak{p}_t).$$

□

In summary, we have the following proposition.

Proposition 4.8. *Let $t = \frac{m+\sqrt{D}}{4} \in \frac{1}{2}\mathcal{O}_F$ with $|m| < \sqrt{D}$. Then the following is true.*

- (1) *When $|\text{Diff}(W, \frac{t}{\sqrt{D}})| > 1$, we have $a(\frac{t}{\sqrt{D}}, \phi_{\mu_5}) = 0$.*
- (2) *When $\text{Diff}(W, \frac{t}{\sqrt{D}}) = \{\mathfrak{p}\}$ with $\mathfrak{p} \nmid 2$, let $\mathfrak{p}_2$ be the unique prime of F over 2. Then we have*

$$a(\frac{t}{\sqrt{D}}, \phi_{\mu_5}) = -\epsilon(D) L_{\mathfrak{p}_2}(1, \chi_{E/F}) \frac{1 + \text{ord}_{\mathfrak{p}}(t\mathcal{O}_F)}{2} \rho_{E/F}(2t\mathfrak{p}) \log N(\mathfrak{p}).$$

Here

$$\epsilon(D) = \begin{cases} 2 & \text{if } D \equiv 1 \pmod{8}, \\ 1 & \text{if } D \equiv 5 \pmod{8}, \\ \frac{1}{2} & \text{if } D \equiv 0 \pmod{4}. \end{cases}$$

- (3) *When $\text{Diff}(W, \frac{t}{\sqrt{D}}) = \{\mathfrak{p}\}$ with $\mathfrak{p}|2$, we have $d_1 \equiv d_2 \equiv 5 \pmod{8}$, and $2\mathcal{O}_F = \mathfrak{p}\tilde{\mathfrak{p}}$ where $\text{ord}_{\mathfrak{p}}(2t) > 0$ is odd and $\text{ord}_{\tilde{\mathfrak{p}}}(2t) = 0$. In this case, we have*

$$a(\frac{t}{\sqrt{D}}, \phi_{\mu_5}) = -\epsilon(D) L_{\tilde{\mathfrak{p}}}(1, \chi_{E/F}) \left[\frac{1 + \text{ord}_{\mathfrak{p}}(2t\mathfrak{p}^{-1})}{2} + \frac{1}{3} \right] \rho_{E/F}(2t\mathfrak{p}) \log N(\mathfrak{p}),$$

with $\epsilon(D) = 2$ being as in (2).

Here is a variant of the proposition, which will be used to prove the main theorem.

Proposition 4.9. *Let $t = \frac{m+\sqrt{D}}{4} \in \frac{1}{2}\mathcal{O}_F$ with $|m| < \sqrt{D}$. Then the following is true.*

- (1) *When $F/\mathbb{Q}$ is split at 2 (i.e., $D \equiv 1 \pmod{8}$), write $2\mathcal{O}_F = \mathfrak{p}_t\tilde{\mathfrak{p}}_t$ with $t \in \mathcal{O}_{F, \mathfrak{p}_t}$ (so $t \notin \mathcal{O}_{F, \tilde{\mathfrak{p}}_t}$), one has*

$$a(\frac{t}{\sqrt{D}}, \phi_{\mu_5}) = -\epsilon(D) L_{\tilde{\mathfrak{p}}_t}(1, \chi_{E/F}) \sum_{\mathfrak{p} \text{ inert in } E/F} \left(\frac{1 + \text{ord}_{\mathfrak{p}}(2t\mathfrak{p}_t^{-2})}{2} + \frac{\text{ord}_{\mathfrak{p}_t}(\mathfrak{p})}{3} \right) \rho_{E/F}(2t\mathfrak{p}\mathfrak{p}_t^{-2}) \log N(\mathfrak{p}).$$

- (2) *When $F/\mathbb{Q}$ is non-split at 2, let $\mathfrak{p}_t = \mathcal{O}_{F,2}$ and $\tilde{\mathfrak{p}}_t$ be the unique prime of F over 2. Then*

$$a(\frac{t}{\sqrt{D}}, \phi_{\mu_5}) = -\epsilon(D) L_{\tilde{\mathfrak{p}}_t}(1, \chi_{E/F}) \sum_{\mathfrak{p} \text{ inert in } E/F} \frac{1 + \text{ord}_{\mathfrak{p}}(2t\mathfrak{p}_t^{-2})}{2} \rho_{E/F}(2t\mathfrak{p}\mathfrak{p}_t^{-2}) \log N(\mathfrak{p}).$$

4.5. **Computing $a_0(\phi_\mu)$.** Let $\psi'(x) = \psi_F(\frac{x}{\sqrt{D}})$ and $W' = E$ with $Q'_F(x) = 2x\bar{x}$ as before. Proposition 5.7 (3) and Remark 4.3 imply

$$(4.10) \quad a_0(\phi_\mu) = -\Lambda(0, \chi) \frac{d}{ds} \prod_{p|2} \tilde{W}_{0,p}^{\psi'}(0, \phi_\mu)|_{s=0},$$

where

$$(4.11) \quad \tilde{W}_{0,p}^{\psi'}(s, \phi_\mu) = \frac{L_p(s+1, \chi)}{L_p(s, \chi)} |2|_p \frac{W_{0,p}^{\psi'}(s, \phi_\mu)}{\gamma(W'_p) \text{vol}(\mathcal{O}_{E_p}, d_{Q'_F} x, \psi')}.$$

By Corollary 5.6 one has $a_0(\phi_{\mu_6}) = 0$. For other μ_j s, it is complicated and we need to do it on a case-by-case basis. The main tools are Corollary 5.6 and Proposition 5.7. Let

$$(4.12) \quad b_i = \text{ord}_2\left(\frac{d_i^2 - d_i}{4}\right) = \text{ord}_2(e_{i+1}\bar{e}_{i+1}) = \begin{cases} 0 & \text{if } d_i \equiv 4, 5 \pmod{8}, \\ 1 & \text{if } d_i \equiv 0 \pmod{8}, \\ \text{ord}_2(d_i - 1) - 2 \geq 1 & \text{if } d_i \equiv 1 \pmod{8}, \end{cases}$$

and

$$c_1 = \text{ord}_2((1 + e_2)(1 + \bar{e}_2)) = \text{ord}_2(1 - d_1 + \frac{d_1^2 - d_1}{4}) = \begin{cases} 0 & \text{if } d_1 \equiv 0, 5 \pmod{8}, \\ 1 & \text{if } d_1 \equiv 4 \pmod{8}, \\ \geq 2 & \text{if } d_1 \equiv 1 \pmod{16}, \\ 1 & \text{if } d_1 \equiv 9 \pmod{16}. \end{cases}$$

$$c_2 = \text{ord}_2((1 + e_3)(1 + \bar{e}_3)) = \text{ord}_2(1 + d_2 + \frac{d_2^2 - d_2}{4}) = \begin{cases} 0 & \text{if } d_2 \equiv 0, 5 \pmod{8}, \\ 1 & \text{if } d_2 \equiv 4 \pmod{8}, \\ 1 & \text{if } d_2 \equiv 1 \pmod{16}, \\ \geq 2 & \text{if } d_2 \equiv 9 \pmod{16}. \end{cases}$$

We denote $\mathfrak{p}$ (or $\mathfrak{p}_i$) for primes of F above 2, and $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ (or $\mathfrak{B}_i$ and $\bar{\mathfrak{B}}_i$) for primes of E above $\mathfrak{p}$ (or $\mathfrak{p}_i$). First we have the following table, where $a_{\mathfrak{p}}(\mu, 0) = \text{ord}_{\mathfrak{p}}(2\mu\bar{\mu})$.

TABLE 2. When does $a_{\mathfrak{p}}(\mu, 0) = \text{ord}_{\mathfrak{p}}(2\mu\bar{\mu}) \geq 0$?

$d_1 \pmod{8}$	$d_2 \pmod{8}$	$a_{\mathfrak{p}}(\mu, 0) < 0$	$a_{\mathfrak{p}}(\mu, 0) \geq 0$
1	0	μ_7	$\mu_1, \mu_2, \mu_3, \mu_4, \mu_9, \mu_{10}, \mu_{12}$
1	1		$\mu_1, \mu_2, \mu_3, \mu_4, \mu_7, \mu_9, \mu_{10}, \mu_{12}$
1	4	μ_4	$\mu_1, \mu_2, \mu_3, \mu_7, \mu_9, \mu_{10}, \mu_{12}$
1	5	μ_4, μ_7	$\mu_1, \mu_2, \mu_3, \mu_9, \mu_{10}, \mu_{12}$
5	0	$\mu_1, \mu_3, \mu_7, \mu_9, \mu_{10}$	μ_2, μ_4, μ_{12}
5	4	$\mu_2, \mu_3, \mu_4, \mu_{10}, \mu_{12}$	μ_1, μ_7, μ_9
5	5	$\mu_1, \mu_2, \mu_3, \mu_4, \mu_7, \mu_9, \mu_{10}, \mu_{12}$	

Lemma 4.10. Recall that $(d_1, d_2) = 1$. Then the following is true.

- (1) When $d_1 \equiv d_2 \equiv 5 \pmod{8}$, one has $a_0(\phi_\mu) = 0$ for $\mu = \mu_1, \mu_2, \mu_3, \mu_4, \mu_7, \mu_9, \mu_{10}, \mu_{12}$.
(2) When $d_1 \equiv 5 \pmod{8}$, and $d_2 \equiv 4 \pmod{8}$, we have

$$a_0(\phi_\mu) = \begin{cases} 0 & \text{if } \mu_2, \mu_3, \mu_4, \mu_{10}, \mu_{12}, \\ -\frac{\Lambda(0, \chi)}{3} \log 2 & \text{if } \mu = \mu_1, \mu_7, \mu_9. \end{cases}$$

- (3) When $d_1 \equiv 5 \pmod{8}$ and $d_2 \equiv 0 \pmod{8}$, we have

$$a_0(\phi_\mu) = \begin{cases} 0 & \text{if } \mu_1, \mu_3, \mu_7, \mu_9, \mu_{10}, \\ -\frac{\Lambda(0, \chi)}{3} \log 2 & \text{if } \mu = \mu_2, \mu_4, \mu_{12}. \end{cases}$$

Proof. Claim (1) follows from Table 2 and Corollary 5.6. For (2) and (3), let $\mathfrak{p}$ be the unique ramified prime of F above 2 and let $\mathfrak{B} = \mathfrak{p}\mathcal{O}_E$ be the unique inert prime of E above $\mathfrak{p}$.

When $d_2 \equiv 4 \pmod{8}$, one has $d_2 \equiv -4 \pmod{16}$ (as $d_2/4 \equiv -1 \pmod{4}$) and so $c_2 = 1$. As $(1 + e_3)(1 + \bar{e}_3) \in \mathbb{Q}_2$, we have

$$\text{ord}_{\mathfrak{B}}\left(\frac{1}{2}(1 + e_3)\right) = \frac{1}{2} \text{ord}_{\mathfrak{p}}\left(\frac{1}{4}(1 + e_3)(1 + \bar{e}_3)\right) = \text{ord}_2\left(\frac{1}{4}(1 + e_3)(1 + \bar{e}_3)\right) = (c_2 - 2) = -1.$$

So $\mu \notin \mathcal{O}_{\mathfrak{B}}$ and $a_{\mathfrak{p}}(\mu, 0) = 0$ for $\mu = \mu_1, \mu_7, \mu_9$. Now (2) follows from Corollary 5.6.

When $d_2 \equiv 0 \pmod{8}$, one has $a_{\mathfrak{p}}(\mu, 0) = 0$ for $\mu = \mu_2, \mu_4, \mu_{12}$. So (3) follows from Corollary 5.6. □

When $d_1 \equiv 1 \pmod{8}$, and $d_2 \not\equiv 1 \pmod{8}$, let $\mathfrak{p}$ be the unique prime of F over 2, then $\mathfrak{p} = \mathfrak{P}\bar{\mathfrak{P}}$ is split in E . Let $\sqrt{d_1}$ be the unique square root of d_1 in $\mathbb{Q}_2$ with $\sqrt{d_1} \equiv 1 \pmod{4}$, and identify

$$E \subset E_{\mathfrak{p}} = E_{\mathfrak{P}} \times E_{\bar{\mathfrak{P}}} \cong F_{\mathfrak{p}} \times F_{\bar{\mathfrak{p}}}, \quad \mu \mapsto (\mu, \bar{\mu}).$$

Let $o_{E_{\mathfrak{p}}}(\mu) = \min(o_{\mathfrak{p}}(\mu), o_{\mathfrak{p}}(\bar{\mu}))$ as in (5.4). Direct calculation gives the following table:

TABLE 3. $o_{E_{\mathfrak{p}}}(2\mu)$

	$d_2 \equiv 5 \pmod{8}$	$d_2 \equiv 0 \pmod{8}$	$d_2 \equiv 4 \pmod{8}$
$\mu_1 = \frac{1}{2}e_2(1 + e_3)$	0	0	1
$\mu_2 = \frac{1}{2}e_2e_3$	0	1	0
$\mu_3 = \frac{1}{2}e_2$	0	0	0
$\mu_4 = \frac{1}{2}e_3$	0	1	0
$\mu_7 = \frac{1}{2}(1 + e_3)$	0	0	1
$\mu_9 = \frac{1}{2}(1 + e_2)(1 + e_3)$	0	0	1
$\mu_{10} = \frac{1}{2}(1 + e_2)$	0	0	0
$\mu_{12} = \frac{1}{2}e_3(1 + e_2)$	0	1	0

Notice also that $a_{\mathfrak{p}}(\mu, 0) = \text{ord}_{\mathfrak{p}}(2\mu\bar{\mu}) < 0$ for $\mu = \mu_6$ always and for $\mu = \mu_7$ with $d_2 \equiv 0, 5 \pmod{8}$ and for $\mu = \mu_4$ with $d_2 \equiv 4, 5 \pmod{8}$.

Lemma 4.11. *Assume that $d_1 \equiv 1 \pmod{8}$, and $d_2 \not\equiv 1 \pmod{8}$. Then $a_0(\phi_{\mu_6}) = 0$, and*

(1) *When $d_2 \equiv 5 \pmod{8}$, one has*

$$a_0(\phi_{\mu_j}) = \begin{cases} 0 & \text{if } j = 4, 7, \\ -\frac{2\Lambda(0, \chi)}{3} \log 2 & \text{if } j = 1, 2, 3, 9, 10, 12. \end{cases}$$

(2) *When $d_2 \equiv 0 \pmod{8}$, one has*

$$a_0(\phi_{\mu_j}) = \begin{cases} 0 & \text{if } j = 4, 7, \\ -\frac{\Lambda(0, \chi)}{2} \log 2 & \text{if } j = 1, 3, 9, 10 \\ -\Lambda(0, \chi) \log 2 & \text{if } j = 2, 12. \end{cases}$$

(3) *When $d_2 \equiv 4 \pmod{8}$, one has*

$$a_0(\phi_{\mu_j}) = \begin{cases} 0 & \text{if } j = 4, 7 \\ -\frac{\Lambda(0, \chi)}{2} \log 2 & \text{if } j = 2, 3, 10, 12, \\ -\Lambda(0, \chi) \log 2 & \text{if } j = 1, 9. \end{cases}$$

Proof. Let $\mu = \mu_1, \mu_2, \mu_3, \mu_9, \mu_{10}, \mu_{12}$, then $a_p(\mu, 0) \geq 0$. We see that if $o_{E_p}(2\mu) = 0$ then only one of μ and $\bar{\mu}$ belong to $\mathcal{O}_{F_p}$. We can also check directly that if $o_{E_p}(2\mu) = 1$ then $a_p(\mu, 0) \geq 1$, and again only one of μ and $\bar{\mu}$ belong to $\mathcal{O}_{F_p}$. Then using table [3](#), equations [\(4.10\)](#), [\(4.11\)](#) and Corollary [5.6](#) we can get the formulae in the lemma by noticing that 2 is inert in F if $d \equiv 5 \pmod{8}$ and ramified otherwise. The cases $\mu = \mu_4, \mu_7$ can also be dealt with similarly by noting that both μ and $\bar{\mu}$ belong to $\mathcal{O}_{F_p}$ in both cases. $\square$

Finally, we deal with the case $d_1 \equiv d_2 \equiv 1 \pmod{8}$. In this case, we have the following lemma.

Lemma 4.12. *Assume $d_i \equiv 1 \pmod{8}$ with $i = 1, 2$. We have*

$$a_0(\phi_{\mu_j}) = \begin{cases} 0 & \text{if } j = 3, 4, 7, 10, \\ -\Lambda(0, \chi) \log 2 & \text{if } j = 1, 2, 9, 12. \end{cases}$$

Proof. Write

$$2\mathcal{O}_F = \mathfrak{p}_1 \mathfrak{p}_2, \quad \mathfrak{p}_i \mathcal{O}_E = \mathfrak{P}_i \bar{\mathfrak{P}}_i.$$

Let $\sqrt{d_i} \equiv 1 \pmod{4} \in \mathbb{Z}_2$ be the prefixed square roots of d_i and let $\sqrt{D} = -\sqrt{d_1} \sqrt{d_2} \in \mathbb{Z}_2$. We identify $F_{\mathfrak{p}_i}$, $E_{\mathfrak{P}_i}$, and $E_{\bar{\mathfrak{P}}_i}$ with $\mathbb{Q}_2$ as follows:

$$\begin{aligned} F_{\mathfrak{p}_i} &\cong \mathbb{Q}_2, & \sqrt{D} &\mapsto (-1)^{i-1} \sqrt{D}, \\ E_{\mathfrak{P}_i} &\cong \mathbb{Q}_2, & \sqrt{D} &\mapsto (-1)^{i-1} \sqrt{D}, \sqrt{d_i} \mapsto \sqrt{d_i}, \\ E_{\bar{\mathfrak{P}}_i} &\cong \mathbb{Q}_2, & \sqrt{D} &\mapsto (-1)^{i-1} \sqrt{D}, \sqrt{d_i} \mapsto -\sqrt{d_i}. \end{aligned}$$

We also fix the following embedding

$$E \subset E \otimes_{\mathbb{Q}} \mathbb{Q}_2 = E_{\mathfrak{P}_1} \times E_{\bar{\mathfrak{P}}_1} \times E_{\mathfrak{P}_2} \times E_{\bar{\mathfrak{P}}_2} \cong \mathbb{Q}_2^4$$

such that

$$e_2 \mapsto (e_2, \bar{e}_2, \bar{e}_2, e_2), \quad e_3 \mapsto (e_3, \bar{e}_3, e_3, \bar{e}_3).$$

With this identification and direct calculation, we have Table 4

TABLE 4. Valuations of μ — when $d_1, d_2 \equiv 1 \pmod{8}$

μ	$\text{ord}_{\mathfrak{p}_1} \mu$	$\text{ord}_{\bar{\mathfrak{p}}_1} \mu$	$\text{ord}_{\mathfrak{p}_2} \mu$	$\text{ord}_{\bar{\mathfrak{p}}_2} \mu$	$\mu \in \mathcal{O}_{E_{\mathfrak{p}_1}}?$	$\mu \in \mathcal{O}_{E_{\mathfrak{p}_2}}?$
e_2	≥ 1	0	0	≥ 1	Yes	Yes
$1 + e_2$	0	≥ 1	≥ 1	0	Yes	Yes
e_3	0	≥ 1	0	≥ 1	Yes	Yes
$1 + e_3$	≥ 1	0	≥ 1	0	Yes	Yes
$\mu_1 = \frac{1}{2}e_2(1 + e_3)$	≥ 1	-1	≥ 0	≥ 0	No	Yes
$\mu_2 = \frac{1}{2}e_2e_3$	≥ 0	≥ 0	-1	≥ 0	Yes	No
$\mu_3 = \frac{1}{2}e_2$	≥ 0	-1	-1	≥ 0	No	No
$\mu_4 = \frac{1}{2}e_3$	-1	≥ 0	-1	≥ 0	No	No
$\mu_6 = \frac{1}{2}$	-1	-1	-1	-1	No	No
$\mu_7 = \frac{1}{2}(1 + e_3)$	≥ 0	-1	≥ 0	-1	No	No
$\mu_9 = \frac{1}{2}(1 + e_2)(1 + e_3)$	≥ 0	≥ 0	≥ 0	-1	Yes	No
$\mu_{10} = \frac{1}{2}(1 + e_2)$	-1	≥ 0	≥ 0	-1	No	No
$\mu_{12} = \frac{1}{2}e_3(1 + e_2)$	-1	≥ 1	≥ 0	≥ 0	No	Yes

By (4.10) and (4.11), we have

$$a_0(\phi_\mu) = -\Lambda(0, \chi) \frac{d}{ds} \left[\prod_{j=1}^2 \frac{W_{0, \mathfrak{p}_j}^{\psi'}(s, \phi_\mu)}{\gamma(W'_{\mathfrak{p}_j}) \text{vol}(\mathcal{O}_{E_{\mathfrak{p}_j}}, d_2 x)} \right]_{s=0}$$

with

$$\tilde{W}_{0, \mathfrak{p}_j}(s, \phi_\mu) = \frac{1}{2} \frac{1 - 2^{-s}}{1 - 2^{-1-s}} \frac{W_{0, \mathfrak{p}_j}^{\psi'}(s, \phi_\mu)}{\gamma(W'_{\mathfrak{p}_j}) \text{vol}(\mathcal{O}_{E_{\mathfrak{p}_j}}, d_2 x)}.$$

When $\mu \notin \mathcal{O}_{E_{\mathfrak{p}_j}}$, $W_{0, \mathfrak{p}_j}^{\psi'}(s, \phi_\mu)$ is a polynomial of 2^{-s} , and so

$$\tilde{W}_{0, \mathfrak{p}_j}(0, \phi_\mu) = 0$$

and

$$\tilde{W}'_{0, \mathfrak{p}_j}(0, \phi_\mu) = \frac{W_{0, \mathfrak{p}_j}^{\psi'}(0, \phi_\mu)}{\gamma(W'_{\mathfrak{p}_j}) \text{vol}(\mathcal{O}_{E_{\mathfrak{p}_j}}, d_2 x)} \log 2.$$

This implies that $a_0(\phi_\mu) = 0$ for $\mu = \mu_3, \mu_4, \mu_7, \mu_{10}$.

When $\mu \in \mathcal{O}_{E_{\mathfrak{p}_j}}$, $\phi_\mu = \phi_0$, one has by Proposition 5.7(3)

$$\tilde{W}_{0, \mathfrak{p}_j}(s, \phi_\mu) = 1 + \frac{1}{2}(1 - 2^{-s}) \cdot (\text{polynomial of } 2^{-s}).$$

Thus $\tilde{W}_{0, \mathfrak{p}_j}(0, \phi_\mu) = 1$.

For $\mu = \mu_2$, the table shows that $\mu \in \mathcal{O}_{E_{p_1}}$ and $\mu \notin \mathcal{O}_{E_{p_2}}$, so we have

$$a_0(\phi_\mu) = -\Lambda(0, \chi) \tilde{W}_{0, p_1}(0, \chi_{\mu_2}) \tilde{W}'_{0, p_2}(0, \phi_{\mu_2}) = -\Lambda(0, \chi) \log 2.$$

The cases for μ_1, μ_9, μ_{12} are similar and left to the reader. □

In summary, we have the following lemma which is needed to prove Theorem [1.4](#) and Proposition [1.3](#)

Proposition 4.13. *The following are true.*

(1) *One has*

$$\begin{aligned} & 8a_0(\phi_{\mu_2}) - 8a_0(\phi_{\mu_3}) - 8a_0(\phi_{\mu_4}) - 8a_0(\phi_{\mu_6}) - 8a_0(\phi_{\mu_7}) + 8a_0(\phi_{\mu_9}) - 8a_0(\phi_{\mu_{10}}) \\ &= -4\Lambda(0, \chi) \log 2 \begin{cases} 4 & \text{if } (d_1, d_2) \equiv (1 \pmod{8}, 1 \pmod{8}), \\ 1 & \text{if } (d_1, d_2) \equiv (1 \pmod{8}, 0 \pmod{4}) \text{ or } (0 \pmod{4}, 1 \pmod{8}), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(2) *One has*

$$\begin{aligned} & a_0(\phi_{\mu_2}) - a_0(\phi_{\mu_3}) + a_0(\phi_{\mu_4}) - a_0(\phi_{\mu_6}) - a_0(\phi_{\mu_{10}}) + a_0(\phi_{\mu_{12}}) \\ &= \Lambda(0, \chi) \log 2 \begin{cases} 0 & \text{if } d_1 \equiv 5 \pmod{8}, d_2 \equiv 4, 5 \pmod{8}, \\ -1 & \text{if } d_1 \equiv 5 \pmod{8}, d_2 \equiv 0 \pmod{8}, \\ -2 & \text{if } d_1 \equiv d_2 \equiv 1 \pmod{8}. \end{cases} \end{aligned}$$

(3) *One has*

$$\begin{aligned} & a_0(\phi_{\mu_1}) - a_0(\phi_{\mu_3}) - a_0(\phi_{\mu_6}) + a_0(\phi_{\mu_7}) + a_0(\phi_{\mu_9}) - a_0(\phi_{\mu_{10}}) \\ &= \Lambda(0, \chi) \log 2 \begin{cases} 0 & \text{if } d_1 \equiv 5 \pmod{8}, d_2 \equiv 0, 5 \pmod{8}, \\ -1 & \text{if } d_1 \equiv 5 \pmod{8}, d_2 \equiv 4 \pmod{8}, \\ -2 & \text{if } d_1 \equiv d_2 \equiv 1 \pmod{8}. \end{cases} \end{aligned}$$

Proof of Theorem [1.4](#) Now Theorem [1.4](#) follows from Propositions [4.5](#), [4.9](#) and [4.13](#)(1), and the fact that

$$\Lambda(0, \chi) = \Lambda(0, \chi_{\mathbf{k}_1/\mathbb{Q}}) \Lambda(0, \chi_{\mathbf{k}_2/\mathbb{Q}}) = \frac{4h_1h_2}{w_1w_2}.$$

Proof of Proposition [1.3](#) Take $d_2 = d$, $\tau_2 = \tau = \frac{d+\sqrt{d}}{2}$. Choose some $d_1 < -4$ with $(d_1, d_2) = 1$, then we have by Corollary [3.4](#) and Theorem [4.4](#) (recall $\lambda_0 = \lambda(\frac{d+\sqrt{d}}{2})$) that

$$\log |N(\lambda_0)| = -\frac{r(d_1)r(d_2)}{4\delta(d_2)h(\mathbf{k}_1, 2)} 8(a_0(\phi_{\mu_2}) - a_0(\phi_{\mu_3}) + a_0(\phi_{\mu_4}) - a_0(\phi_{\mu_6}) - a_0(\phi_{\mu_{10}}) + a_0(\phi_{\mu_{12}})),$$

Now applying Proposition 4.13 (2), we obtain via a little calculation

$$\log |N(\lambda_0)| = \begin{cases} 0 & \text{if } d \equiv 4, 5 \pmod{8}, \\ 2h & \text{if } d \equiv 0 \pmod{8}, \\ 4h & \text{if } d \equiv 1 \pmod{8} \end{cases}$$

as claimed. The same argument with

$$\log |N(1-\lambda_0)| = -\frac{r(d_1)r(d_2)}{4\delta(d_2)h(\mathbf{k}_1, 2)} 8(a_0(\phi_{\mu_1}) - a_0(\phi_{\mu_3}) - a_0(\phi_{\mu_6}) + a_0(\phi_{\mu_7}) + a_0(\phi_{\mu_9}) - a_0(\phi_{\mu_{10}}))$$

gives the desired formula for $|N(1-\lambda_0)|$. This proves Proposition 1.3

4.6. Proof of Theorem 1.6. In this subsection, we will study the set $\mathcal{N}(d_1, d_2)$ and prove Theorem 1.6. For an imaginary quadratic field $\mathbf{k} = \mathbb{Q}(\sqrt{d})$, set $\tau^0 = \frac{d+\sqrt{d}}{2}$. We also write $\tau_1^0 = \frac{d_1+\sqrt{d_1}}{2}$ and $\tau_2^0 = \frac{d_2+\sqrt{d_2}}{2}$. Denote

$$\lambda_1 = \lambda, \lambda_2 = 1 - \lambda, \lambda_3 = \frac{1}{\lambda}, \lambda_4 = \frac{\lambda}{\lambda - 1}, \lambda_5 = \frac{\lambda - 1}{\lambda}, \text{ and } \lambda_6 = \frac{1}{1 - \lambda}.$$

Then the values $\lambda(\tau)$, $\tau \in \text{CM}(\mathbf{k}, 2)$, are determined up to Galois conjugation by $\lambda_i(\tau^0)$, $1 \leq i \leq 6$.

Lemma 4.14. *Let $a(d) = h(\mathbf{k}, 2)$ or $2h(\mathbf{k}, 2)$ depending on whether $4|d$ or not. Then*

$$\begin{aligned} \mathcal{N}(d_1, d_2) = & \cup \{ |N(\lambda(\tau_1^0) - \lambda_i(\tau_2^0))| : 1 \leq i \leq 6 \} \\ & \cup \left\{ \left| \frac{N(\lambda(\tau_1^0) - \lambda_i(\tau_2^0))}{N(\lambda(\tau_1^0)^{a(d_2)}) N(\lambda_i(\tau_2^0)^{a(d_1)})} \right| : 1 \leq i \leq 6 \right\} \\ & \cup \left\{ \left| \frac{N(\lambda(\tau_1^0) - \lambda_i(\tau_2^0))}{N((1 - \lambda(\tau_1^0))^{a(d_2)}) N((1 - \lambda_i(\tau_2^0))^{a(d_1)})} \right| : 1 \leq i \leq 6 \right\}. \end{aligned}$$

Proof. For $\tau_l \in \text{CM}(\mathbf{k}_l, 2)$ with $l = 1, 2$, we may assume (up to Galois conjugation) $\lambda(\tau_1) = \lambda_i(\tau_1^0)$ and $\lambda(\tau_2) = \lambda_i(\tau_2')$ for some i . Simple computation gives

$$\lambda(\tau_1) - \lambda(\tau_2) = \lambda_i(\tau_1^0) - \lambda_i(\tau_2') = \pm \begin{cases} \lambda(\tau_1^0) - \lambda(\tau_2') & \text{if } i = 1, 2, \\ \frac{\lambda(\tau_1^0) - \lambda(\tau_2')}{\lambda(\tau_1^0)\lambda(\tau_2')} & \text{if } i = 3, 5, \\ \frac{\lambda(\tau_1^0) - \lambda(\tau_2')}{(1 - \lambda(\tau_1^0))(1 - \lambda(\tau_2'))} & \text{if } i = 4, 6. \end{cases}$$

Now write $\lambda(\tau_2') = \lambda_i(\tau_2^0)$ for some i . Taking norm on both sides of the above identity proves the lemma. $\square$

Let

$$\mathcal{N}_0(d_1, d_2) = \{ |N(\lambda(\tau_1^0) - \lambda_i(\tau_2^0))| : 1 \leq i \leq 6 \} = \{ |N(\lambda(\tau_1^0) - \lambda(\tau_2))| : \tau_2 \in \text{CM}(\mathbf{k}_2, 2) \}.$$

Then $|\mathcal{N}_0(d_1, d_2)| = 1$ when $d_2 \equiv 5 \pmod{8}$ and $|\mathcal{N}_0(d_1, d_2)| \leq 3$ otherwise by Proposition 2.3

Lemma 4.15. *Assume $d_1 \equiv 1 \pmod{8}$ and $d_2 \not\equiv 5 \pmod{8}$. Then $|\mathcal{N}_0(d_1, d_2)| = 2$. More precisely, write $\mathcal{N}_0(d_1, d_2) = \{\alpha, \beta, \gamma\}$ with*

$$\begin{aligned}\alpha &= |\mathrm{N}(\lambda(\tau_1^0) - \lambda(\tau_2^0))| = |\mathrm{N}(\lambda(\tau_1^0) - \lambda_2(\tau_2^0))|, \\ \beta &= |\mathrm{N}(\lambda(\tau_1^0) - \lambda_3(\tau_2^0))| = |\mathrm{N}(\lambda(\tau_1^0) - \lambda_5(\tau_2^0))|, \\ \gamma &= |\mathrm{N}(\lambda(\tau_1^0) - \lambda_4(\tau_2^0))| = |\mathrm{N}(\lambda(\tau_1^0) - \lambda_6(\tau_2^0))|.\end{aligned}$$

Then the following are true.

TABLE 5. α, β, γ

$d_2 \pmod{8}$	4	0	1
	$\alpha = \beta$	$\alpha = \gamma$	$\beta = \gamma$

TABLE 6. 2-parts in α, β, γ

$d_2 \pmod{8}$	4 or 0	1
$\mathrm{ord}_2(\alpha)$	$2h_1h_2$	$8h_1h_2$
$\mathrm{ord}_2(\tilde{\alpha})$	$-4h_1h_2$	$-4h_1h_2$

Here $\tilde{\alpha}$ is the element of $\{\beta, \gamma\}$ that is different from α .

Proof. Proposition 1.2 asserts that $\lambda(\tau_1^0)$ and $1 - \lambda(\tau_1^0)$ are Galois conjugates of each other. So

$$|\mathrm{N}(\lambda(\tau_1^0) - \lambda_i(\tau_2^0))| = |\mathrm{N}(\lambda(\tau_1^0) - (1 - \lambda_i(\tau_2^0)))|,$$

which justify the two formulas for α, β and γ . Table 5 also follows directly from Proposition 1.2

Notice that $\lambda_i(\tau_2^0)$, $1 \leq i \leq 6$, are the six roots of $f(\lambda, j(\tau_2^0))$ in equation (1.1). So we have

$$f(\lambda, j(\tau_2^0)) = \prod_{i=1}^6 (\lambda - \lambda_i(\tau_2^0)).$$

In particular,

$$f(\lambda(\tau_1^0), j(\tau_2^0)) = \prod_{i=1}^6 (\lambda(\tau_1^0) - \lambda_i(\tau_2^0)).$$

On the other hand, $\lambda(\tau_1^0)$ is also a root of $f(\lambda, j(\tau_1^0))$. Therefore,

$$\begin{aligned}(4.13) \quad 256 \prod_{i=1}^6 (\lambda(\tau_1^0) - \lambda_i(\tau_2^0)) &= 256(f(\lambda(\tau_1^0), j(\tau_2^0)) - f(\lambda(\tau_1^0), j(\tau_1^0))) \\ &= (j(\tau_1^0) - j(\tau_2^0))\lambda(\tau_1^0)^2(1 - \lambda(\tau_1^0))^2.\end{aligned}$$

Let $H(\mathbf{k}_i, 2)$ be the ray class field of $\mathbb{Q}(\sqrt{d_i})$ of modulus 2, $i = 1, 2$. Then we have $[H(\mathbf{k}_i, 2) : \mathbb{Q}] = 2h_i$ by Lemma 2.1 and we also know that $H(\mathbf{k}_1, 2)$ and $H(\mathbf{k}_2, 2)$ are

disjoint by Lemma 4.1. Taking norms in $H(\mathbf{k}_1, 2)H(\mathbf{k}_2, 2)/\mathbb{Q}$ on both sides of equation (4.13), we have

$$256^{4h_1h_2}(\alpha\beta\gamma)^2 = (\mathrm{N}(j(\tau_1^0) - j(\tau_2^0)))^4 \mathrm{N}(\lambda(\tau_1^0))^{4h_2} \mathrm{N}((1 - \lambda(\tau_1^0)))^{4h_2}.$$

Applying Proposition 1.3 we obtain

$$(4.14) \quad \mathrm{N}(j(\tau_1^0) - j(\tau_2^0))^2 = \alpha\beta\gamma.$$

By the Gross-Zagier singular moduli formula ([GZ85], [Dor88], or [YY Theorem 1.1]), we have $\mathrm{ord}_2(\mathrm{N}(j(\tau_1^0) - j(\tau_2^0))) = 0$. On the other hand, Theorem 1.4 implies

$$\mathrm{ord}_2(\alpha) = \begin{cases} 8h_1h_2 & \text{if } d_2 \equiv 1 \pmod{8}, \\ 2h_1h_2 & \text{if } d_2 \equiv 0 \pmod{4}. \end{cases}$$

Now Table 6 follows from Table 5 and (4.14). $\square$

Proof of Theorem 1.6: Now we are ready to prove Theorem 1.6. If one of the $d_i \equiv 5 \pmod{8}$, for example, we assume $d_2 \equiv 5 \pmod{8}$. Then Lemma 4.14 and Proposition 1.3 imply

$$\mathcal{N}(d_1, d_2) = \left\{ \alpha, \frac{\alpha}{\mathrm{N}(\lambda(\tau_1^0))^{a(d_2)}}, \frac{\alpha}{\mathrm{N}(1 - \lambda(\tau_1^0))^{a(d_2)}} \right\}.$$

Proposition 1.3 shows that $|\mathcal{N}(d_1, d_2)| = 1$ or 2 depending on whether $d_1 \equiv 5 \pmod{8}$ or not.

Now assume that $d_i \not\equiv 5 \pmod{8}$. Then we may assume $d_1 \equiv 1 \pmod{8}$ as $(d_1, d_2) = 1$. Lemmas 4.14, 4.15 and Proposition 1.3 imply

$$\mathcal{N}(d_1, d_2) = \left\{ \alpha, \frac{2^{-8h_1h_2}\alpha}{\mathrm{N}(\lambda(\tau_2^0))^{2h_1}}, \frac{2^{-8h_1h_2}\alpha}{\mathrm{N}(1 - \lambda(\tau_2^0))^{2h_1}}, \tilde{\alpha}, \frac{2^{-8h_1h_2}\tilde{\alpha}}{\mathrm{N}(\lambda_i(\tau_2^0))^{2h_1}}, \frac{2^{-8h_1h_2}\tilde{\alpha}}{\mathrm{N}(1 - \lambda_i(\tau_2^0))^{2h_1}} \right\},$$

where $\tilde{\alpha} = \mathrm{N}(\lambda(\tau_1^0) - \lambda_i(\tau_2^0)) \neq \alpha$ for some i as in Lemma 4.15

When $d_2 \equiv 1 \pmod{8}$, then Proposition 1.3 and Lemma 4.15 show that

$$\mathcal{N}(d_1, d_2) = \{ \alpha, 2^{-16h_1h_2}\alpha, \tilde{\alpha}, 2^{-8h_1h_2}\tilde{\alpha} \}$$

has cardinality 4 as their 2-adic valuations are distinct: $8h_1h_2, -8h_1h_2, -4h_1h_2, -12h_1h_2$.

Similarly one has for $d_2 \equiv 4, 0 \pmod{8}$,

$$\mathcal{N}(d_1, d_2) = \{ \alpha, 2^{-8h_1h_2}\alpha, 2^{-12h_1h_2}\alpha, \tilde{\alpha}, 2^{-4h_1h_2}\tilde{\alpha} \}$$

which has cardinality 5. This proves the theorem.

5. APPENDIX A: EXPLICIT FORMULAS FOR WHITTAKER FUNCTIONS

Let F be a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$ and a uniformizer $\pi = \pi_F$. We denote by $|\cdot|$ the valuation of F . Let E be a quadratic étale extension of F , including $F \times F$, with ring of integers $\mathcal{O}_E$ and a uniformizer π_E , and let $\chi = \chi_{E/F}$ be the associated quadratic character of $F^\times$. Let ψ be an unramified additive character of F , and let $\psi_E = \psi \circ \mathrm{tr}_{E/F}$. Fix $0 \neq \beta \in \mathcal{O}_F$, and let $W = W_\beta = E$ with F -quadratic form

$Q_\beta(x) = \beta x \bar{x}$. Let $\omega = \omega_{\beta, \psi}$ be the Weil representation of $G = \mathrm{SL}_2(F)$ on $S(W) = S(E)$ associated to the reductive dual pair $(\mathcal{O}(W), \mathrm{SL}_2)$, and let

$$(5.1) \quad \lambda : S(E) \rightarrow I(0, \chi), \quad \lambda(\phi)(g) = \omega(g)\phi(0).$$

Let $\Phi_\phi \in I(s, \chi)$ be the associated standard section, given by

$$\Phi_\phi(g, s) = \lambda(\phi)(g)|a(g)|^s,$$

where $|a(g)| = |a|$ if $g = n(b)m(a)k$ with $b \in F$, $a \in F^\times$, and $k \in \mathrm{SL}_2(\mathcal{O}_F)$. Recall that

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Whittaker function of Φ_ϕ (and thus ϕ) is defined to be

$$(5.2) \quad W_t(g, s, \phi) = \int_F \Phi_\phi(w n(b)g, s) \psi(-tb) db,$$

where db is the self-dual Haar measure on F with respect to ψ , i.e., $\mathrm{vol}(\mathcal{O}_F) = 1$. The purpose of this appendix is to compute the Whittaker function

$$W_t(s, \phi) = W_t(1, s, \phi)$$

explicitly in the general case, which is of independent interest in addition to the applications in Section 4. Various special cases have been appeared in [Yan05], [HY12, Section 4.6], and [KY10]. Notice that

$$S(E) = \bigcup_{L \subset \mathcal{O}_E, \text{ ideal}} S_L,$$

where S_L is defined in Section 3.1. We can and will assume $\phi = \phi_\mu = \mathrm{Char}(\mu + L)$ in the following. Via shifting, we can and will further assume

$$(5.3) \quad L = \mathcal{O}_E, \quad Q_\beta(x) = \beta x \bar{x}.$$

So

$$L' = \{x \in E : \psi((x, y)) = 1, \text{ for all } y \in \mathcal{O}_E\} = (\beta \partial_{E/F})^{-1} \mathcal{O}_E.$$

We denote $o(x) = \mathrm{ord}_\pi x$ for $x \in F$, and

$$(5.4) \quad o_E(\mu) = \begin{cases} \mathrm{ord}_{\pi_E}(\mu) & \text{if } E/F \text{ non-split,} \\ \min(o(\mu_1), o(\mu_2)) & \text{if } E = F \times F, \mu = (\mu_1, \mu_2). \end{cases}$$

We also write $f = \mathrm{ord}_{\pi_E}(\partial_{E/F})$ and $e = e(E/F) = 1$, or 2 for the ramification index. Define the generalized Gauss integral for $\mu \in E/\mathcal{O}_E$ and $b \in F^\times$ to be

$$(5.5) \quad I_\mu(b) = \int_{\mu + \mathcal{O}_E} \psi(bx \bar{x}) dx,$$

where dx is the standard Haar measure on E with $\mathrm{vol}(\mathcal{O}_E, dx) = 1$. Let $d_\beta x$ be the self-dual Haar measure on $W_\beta = E$ with respect to $\psi((x, y)) = \psi(\mathrm{tr}_{E/F} \beta x \bar{y})$. Then $d_\beta x = |\beta| |d_{E/F}|^{\frac{1}{2}} dx$, i.e. $\mathrm{vol}(\mathcal{O}_E, d_\beta x) = |\beta| |d_{E/F}|^{\frac{1}{2}}$.

Proposition 5.1. *Let ψ be an unramified additive character of F , and $W_\beta = E$ with F -quadratic form $Q_\beta(z) = \beta z \bar{z}$ with $0 \neq \beta \in \mathcal{O}_F$. Let $L = \text{Char}(\mathcal{O}_E)$ and $\mu \in L'$. Then*

$$(5.6) \quad \frac{W_t(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \sum_{n=0}^{\infty} J_\mu(n) q^{n(1-s)},$$

where $\gamma(\beta) = \gamma(W_\beta)$ is the local Weil index of W_β and

$$(5.7) \quad J_\mu(n) = \begin{cases} \int_{\mathcal{O}_F} I_\mu(b\beta) \psi(-tb) db & \text{if } n = 0, \\ \int_{\mathcal{O}_F^\times} I_\mu(b\beta\pi^{-n}) \psi(-tb\pi^{-n}) db & \text{if } n > 0. \end{cases}$$

Moreover,

$$J_\mu(0) = \text{Char}(\mathcal{O}_F)(Q_\beta(\mu) - t).$$

Proof. Since

$$|a(w\eta(b))| = \begin{cases} 1 & \text{if } b \in \mathcal{O}_F, \\ |b|^{-1} & \text{if } b \notin \mathcal{O}_F, \end{cases}$$

one has by definition,

$$\begin{aligned} \frac{W_t(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} &= \frac{1}{\text{vol}(\mathcal{O}_E, d_\beta x)} \int_F \int_E \psi(bQ_\beta(x)) \phi_\mu(x) d_\beta x |a(w\eta(b))|^s \psi(-tb) db \\ &= \int_F I_\mu(b\beta) \psi(-tb) |a(w\eta(b))|^s db \\ &= \sum_{n=0}^{\infty} J_\mu(n) q^{n(1-s)}. \end{aligned}$$

Next,

$$\begin{aligned} J_\mu(0) &= \int_{\mu + \mathcal{O}_E} dx \int_{\mathcal{O}_F} \psi(b(Q_\beta(x) - t)) db \\ &= \int_{\mu + \mathcal{O}_E} \text{Char}(\mathcal{O}_F)(Q_\beta(x) - t) dx \\ &= \text{Char}(\mathcal{O}_F)(Q_\beta(\mu) - t) \end{aligned}$$

as claimed. □

Lemma 5.2. *Let $\mu \in E/\mathcal{O}_E$ and $b \in F^\times$.*

(1) *When $\mu = 0$, one has*

$$I_0(b) = \begin{cases} 1 & \text{if } b \in \mathcal{O}_F, \\ \chi(b)|b|^{-1} & \text{if } b \notin \mathcal{O}_F, \text{ } E/F \text{ unramified,} \\ 0 & \text{if } -f < o(b) < 0, \text{ } E/F \text{ ramified,} \\ \chi(b)|b|^{-1} q^{-\frac{f}{2}} \gamma(W_1) & \text{if } o(b) \leq -f, \text{ } E/F \text{ ramified.} \end{cases}$$

(2) *When $\mu \notin \mathcal{O}_E$ and E/F is unramified (inert or split), one has*

$$I_\mu(b) = \psi(b\mu\bar{\mu}) \text{Char}(\mathcal{O}_E)(b\mu).$$

(3) When $\mu \notin \mathcal{O}_E$ and E/F is ramified, one has

$$I_\mu(b) = \begin{cases} \psi(b\mu\bar{\mu}) \text{Char}(\partial_{E/F}^{-1})(b\mu) & \text{if } b \in \mathcal{O}_F, \\ |b|^{-1} \sum_{a \in \mathcal{O}_E/\pi_E^m} \psi(b(\mu+a)(\overline{\mu+a})) & \text{if } 0 < m \leq o_E(\mu) + f, \\ 0 & \text{if } m > o_E(\mu) + f. \end{cases}$$

Here $m = -o(b) > 0$.

Proof. We first consider the case $\mu = 0$. The case $E = F \times F$ is trivial. Assume that E is a quadratic field extension of F . Then [HY12, Lemma 4.6.1] implies

$$I_0(b) = \int_{\mathcal{O}_E} \psi(bz\bar{z})dz = C \int_{\mathcal{O}_F} \psi(bx)(1 + \chi(x))dx,$$

with $C = 1$ or $\frac{1}{2}(1 + q^{-1})$ depending on whether E/F is ramified or not. When E/F is ramified, the calculation in [HY12, Page 60] gives (1) for $\mu = 0$. When E/F is inert, χ is unramified and $\chi(\pi) = -1$. For $b \notin \mathcal{O}_F$, write $m = -o(b) > 0$, then one has

$$\begin{aligned} \int_{\mathcal{O}_F} \psi(bx)\chi(x)dx &= \sum_{n=0}^{\infty} q^{-n}\chi(\pi)^n \int_{\mathcal{O}_F^\times} \psi(\pi^n bx)dx \\ &= \sum_{n=0}^{\infty} q^{-n}\chi(\pi)^n (\text{Char}(\mathcal{O}_F)(\pi^n b) - q^{-1} \text{Char}(\mathcal{O}_F)(\pi^{n+1}b)) \\ &= \left[(1 - q^{-1}) \sum_{n=m}^{\infty} (-q)^n \right] - q^{-1}(-q)^{-m+1} \\ &= \frac{2}{1 + q^{-1}} \chi(b)|b|^{-1}. \end{aligned}$$

So one has for $b \notin \mathcal{O}_F$,

$$I_0(b) = \chi(b)|b|^{-1}$$

as claimed. For $b \in \mathcal{O}_F$, it is trivial. This takes care the case $\mu \in \mathcal{O}_E$.

Next, we assume that $\mu \notin \mathcal{O}_E$ and E/F is unramified. When $b\mu \in \mathcal{O}_E$, one has $b \in \mathcal{O}_F$ and

$$I_\mu(b) = \psi(b\mu\bar{\mu}) \int_{\mathcal{O}_E} \psi(\text{tr}_{E/F}(b\bar{\mu}z))\psi(bz\bar{z})dz = \psi(b\mu\bar{\mu}) \int_{\mathcal{O}_E} \psi(bz\bar{z})dz = \psi(b\mu\bar{\mu}).$$

When $b\mu \notin \mathcal{O}_E$, and $b \in \mathcal{O}_F$, the same computation gives

$$I_\mu(b) = \psi(b\mu\bar{\mu}) \int_{\mathcal{O}_E} \psi(\text{tr}_{E/F}(b\bar{\mu}z))dz = 0$$

Finally, when $b\mu \notin \mathcal{O}_E$, and $b \notin \mathcal{O}_F$, A substitution $z \mapsto z + b^{-1}u$, $u \in \mathcal{O}_E^\times$, gives

$$I_\mu(b) = \psi(\text{tr}_{E/F}(u\bar{\mu}))I_\mu(b).$$

Since there is $u \in \mathcal{O}_F^\times$ such that $\psi(\text{tr}_{E/F}(u\bar{\mu})) \neq 1$, one sees that $I_\mu(b) = 0$. This proves (2).

Finally, assume that $\mu \notin \mathcal{O}_E$ and E/F is ramified. The case $b \in \mathcal{O}_F$ is the same as in the proof of (2). Now assume $b \notin \mathcal{O}_F$, so $m = -o(b) > 0$.

$$\begin{aligned}
I_\mu(b) &= \sum_{a \in \mathcal{O}_E/\pi_E^m} \int_{\mu+a+\pi_E^m \mathcal{O}_E} \psi(bx\bar{x}) dx \\
&= \sum_{a \in \mathcal{O}_E/\pi_E^m} q_E^{-m} \psi(b(\mu+a)\overline{(\mu+a)}) \int_{\mathcal{O}_E} \psi_E(b\pi_E^m \overline{(\mu+a)}x) dx \\
&= q_E^{-m} \sum_{a \in \mathcal{O}_E/\pi_E^m} \psi(b(\mu+a)\overline{(\mu+a)}) \text{Char}(\partial_{E/F}^{-1})(b\pi_E^m \overline{(\mu+a)}) \\
&= q^{-m} \sum_{a \in \mathcal{O}_E/\pi_E^m} \psi(b(\mu+a)\overline{(\mu+a)}) \text{Char}(\partial_{E/F}^{-1})(b\pi_E^m \bar{\mu}).
\end{aligned}$$

Now (3) is clear. □

We denote

$$(5.8) \quad \alpha(\mu, t) = Q_\beta(\mu) - t = \beta\mu\bar{\mu} - t, \quad a(\mu, t) = o(Q_\beta(\mu) - t) = o(\beta\mu\bar{\mu} - t).$$

Proposition 5.3. *Assume that E/F is unramified and $0 \neq \mu \in E/\mathcal{O}_E$. Then $W_t(s, \phi_\mu) = 0$ unless $\alpha(\mu, t) \in \mathcal{O}_F$, i.e., $a(\mu, t) \geq 0$. Moreover,*

$$\frac{W_t(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \begin{cases} (1 - q^{-s}) \sum_{0 \leq n \leq a(\mu, t)} q^{n(1-s)} & \text{if } 0 \leq a(\mu, t) < o_E(\beta\mu), \\ (1 - q^{-s}) \sum_{0 \leq n < o_E(\beta\mu)} q^{n(1-s)} + q^{o_E(\beta\mu)(1-s)} & \text{if } a(\mu, t) \geq o_E(\beta\mu). \end{cases}$$

In particular, $W_t(0, \phi_\mu) = 0$ if and only if $a(\mu, t) < o_E(\beta\mu)$. In such a case,

$$\frac{W'_t(0, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \begin{cases} (\sum_{0 \leq n \leq a(\mu, t)} q^n) \log q & \text{if } 0 \leq a(\mu, t) < o_E(\beta\mu), \\ 0 & \text{if } a(\mu, t) < 0. \end{cases}$$

Proof. Since E/F is unramified, one has $\partial_{E/F} = \mathcal{O}_E$, $f = 0$ and $\pi_E = \pi$. When $0 < n \leq o(\beta)$, one has for $b \in \mathcal{O}_F^\times$ by Lemma 5.2

$$I_\mu(\beta b \pi^{-n}) = \psi(\pi^{-n} b Q_\beta(\mu)) \text{Char}(\mathcal{O}_E)(b\mu\beta\pi^{-n}).$$

So $J_\mu(n) = 0$ for $n > o_E(\beta\mu)$. When $0 < n \leq o_E(\beta\mu)$, one has

$$\begin{aligned}
J_\mu(n) &= \int_{\mathcal{O}_F^\times} \psi(\pi^{-n} b (Q_\beta(\mu) - t)) db \\
&= \text{Char}(\pi^n \mathcal{O}_F)(Q_\beta(\mu) - t) - q^{-1} \text{Char}(\pi^{n-1} \mathcal{O}_F)(Q_\beta(\mu) - t).
\end{aligned}$$

So

$$\begin{aligned}
\frac{W_t(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} &= \text{Char}(\mathcal{O}_F)(Q_\beta(\mu) - t) + \\
&\sum_{0 < n \leq o_E(\beta\mu)} [\text{Char}(\pi^n \mathcal{O}_F)(Q_\beta(\mu) - t) - q^{-1} \text{Char}(\pi^{n-1} \mathcal{O}_F)(Q_\beta(\mu) - t)] q^{n(1-s)}.
\end{aligned}$$

Now a simple calculation proves the proposition. $\square$

Proposition 5.4. *Assume that E/F is ramified with $f = o_E(\partial_{E/F}) = o(d_{E/F})$ and $0 \neq \mu \in E/\mathcal{O}_E$. Then $W_t(s, \phi_\mu) = 0$ unless $a(\mu, t) \geq 0$. Write $a = o(\beta) + [\frac{1}{2}(o_E(\mu) + f)]$, where $[x]$ is the largest integer less than or equal to x .*

(1) *When $o_E(\mu) + f \leq 0$, i.e., $\mu \notin \pi_E \partial_{E/F}^{-1}$, one has*

$$\frac{W_t(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \begin{cases} (1 - q^{-s}) \sum_{0 \leq n \leq a(\mu, t)} q^{n(1-s)} & \text{if } a(\mu, t) < a, \\ (1 - q^{-s}) \sum_{0 \leq n < a} q^{n(1-s)} + q^{a(1-s)} & \text{if } a(\mu, t) \geq a. \end{cases}$$

(2) *When $o_E(\mu) + f > 0$, i.e., $\mu \in \pi_E \partial_{E/F}^{-1} - \mathcal{O}_E$, one has*

$$\frac{W_t(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = (1 - q^{-s}) \sum_{0 \leq n \leq a(\mu, t)} q^{n(1-s)}$$

if $a(\mu, t) < o(\beta)$ and

$$\begin{aligned} \frac{W_t(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} &= (1 - q^{-s}) \sum_{0 \leq n < o(\beta)} q^{n(1-s)} + q^{o(\beta)(1-s)} \\ &+ \sum_{o(\beta) < n \leq o(\beta) + o_E(\mu) + f} q^{o(\beta)} q^{-ns} \sum_{a \in \mathcal{O}_E / \pi_E^{n-o(\beta)}} \\ &[\text{Char}(\pi^n \mathcal{O}_F)(\alpha(\mu + a, t)) - q^{-1} \text{Char}(\pi^{n-1} \mathcal{O}_F)(\alpha(\mu + a, t))] \end{aligned}$$

if $a(\mu, t) \geq o(\beta)$.

(3) *One has $W_t(0, \phi_\mu) = 0$ if and only if $a(\mu, t) < \min(o(\beta), a)$. In such a case,*

$$\frac{W'_t(0, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \begin{cases} (\sum_{0 \leq n \leq a(\mu, t)} q^n) \log q & \text{if } 0 \leq a(\mu, t) < \min(o(\beta), a), \\ 0 & \text{if } a(\mu, t) < 0. \end{cases}$$

Remark 5.5. When $p \neq 2$, $f = 1$, the second sum in Proposition 5.4(2) does not appear, the resulting formula is the same as the unramified case. In general, $f \leq 1 + o(2)$, the last sum is manageable when $o(2) = \text{ord}_\pi 2$ is small. However, the ramification index f can be arbitrarily large when $F/\mathbb{Q}_2$ has arbitrary large ramification, and the sum can become complicated. We refer to [Yan04, Appendix A] for an explicit way to compute f .

Proof. Assume $o_E(\mu) + f \leq 0$ first. For $b \in \mathcal{O}_F^\times$, Lemma 5.2 implies

$$I_\mu(\beta b \pi^{-n}) = \begin{cases} \psi(Q_\beta(\mu) b \pi^{-n}) & \text{if } n \leq a, \\ 0 & \text{if } n > a. \end{cases}$$

When $n \leq a$, one has then

$$\begin{aligned} J_\mu(\beta b \pi^{-n}) &= \int_{\mathcal{O}_F^\times} \psi(\pi^{-n}(Q_\beta(\mu) - t)b) db \\ &= \begin{cases} 1 - q^{-1} & \text{if } n \leq \min(a(\mu, t), a), \\ -q^{-1} & \text{if } n = a(\mu, t) + 1 \leq a, \\ 0 & \text{if } n > a(\mu, t) + 1. \end{cases} \end{aligned}$$

Applying Proposition [5.1](#) we obtain (1).

Now assume that $o_E(\mu) + f > 0$. When $m = n - o(\beta) \leq 0$, we have with the same calculation

$$J_\mu(n) = \begin{cases} 1 - q^{-1} & \text{if } n \leq \min(a(\mu, t), o(\beta)), \\ -q^{-1} & \text{if } n = a(\mu, t) + 1 \leq o(\beta), \\ 0 & \text{if } n > a(\mu, t) + 1. \end{cases}$$

When $m > 0$, i.e, $n \geq o(\beta)$, Lemma [5.2](#) implies

$$I_\mu(\beta b \pi^{-n}) = q^{-m} \sum_{a \in \mathcal{O}_E / \pi_E^{n-o(\beta)}} \psi(\pi^{-n} b Q_\beta(\mu + a)),$$

and so

$$J_\mu(n) = q^{-m} \sum_{a \in \mathcal{O}_E / \pi_E^{n-o(\beta)}} [\text{Char}(\pi^n \mathcal{O}_F)(Q_\beta(\mu + a) - t) - q^{-1} \text{Char}(\pi^{n-1} \mathcal{O}_F)(Q_\beta(\mu + a) - t)].$$

Now (2) is clear. Claim (3) follows from (1) and (2). $\square$

Corollary 5.6. *Assume $\mu \in E - \mathcal{O}_E$. Then $W_0(s, \phi_\mu) = 0$ unless $a(\mu, 0) = o(\beta \mu \bar{\mu}) \geq 0$. When $a(\mu, 0) \geq 0$, the following is true.*

(1) *When E/F is non-split, one has*

$$\frac{W_0(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = (1 - q^{-s}) \sum_{0 \leq n \leq a(\mu, 0)} q^{n(1-s)}.$$

In particular, $W_0(0, \phi_\mu) = 0$, and

$$\frac{W'_0(0, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \sum_{0 \leq n \leq a(\mu, 0)} q^n \log q.$$

(2) *When $E = F \oplus F$ is split over F and $\mu = (\mu_1, \mu_2) \in (F^\times)^2$, one has*

$$\frac{W_0(s, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \begin{cases} (1 - q^{-s}) \sum_{0 \leq n \leq a(\mu, 0)} q^{n(1-s)} & \text{if } \mu_1, \mu_2 \notin \mathcal{O}_F, \\ (1 - q^{-s}) \sum_{0 \leq n < o_E(\beta \mu)} q^{n(1-s)} + q^{o_E(\beta \mu)(1-s)} & \text{if only one of } \mu_1 \text{ and } \mu_2 \text{ is in } \mathcal{O}_F. \end{cases}$$

In particular

$$\frac{W'_0(0, \phi_\mu)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \begin{cases} \sum_{0 \leq n \leq a(\mu, 0)} q^n \log q & \text{if } \mu_1, \mu_2 \notin \mathcal{O}_F, \\ \sum_{0 \leq n < o_E(\beta \mu)} q^n \log q - q^{o_E(\beta \mu)} \log q & \text{if only one of } \mu_1 \text{ and } \mu_2 \text{ is in } \mathcal{O}_F. \end{cases}$$

Proof. Since $\mu \notin \mathcal{O}_E$, $a(\mu, 0) = o(\beta\mu\bar{\mu}) < o_E(\beta\mu)$ when E/F is inert, and $a(\mu, 0) < \min(o(\beta), a)$ when E/F is ramified. This implies (1) by Propositions 5.3 and 5.4. The first part of (2) is the same. On the other hand, if one of μ_i , say, $\mu_1 \in \mathcal{O}_F$, one has

$$a(\mu, 0) = o(\beta\mu_1\mu_2) \geq o(\beta\mu_2) = o_E(\beta\mu).$$

Proposition 5.3 implies this case too. □

The case $\mu = 0$ is already computed in [HY12, Section 4.6] using a slightly different method. There are some minor errors there. We record the correct results here for the convenience of the reader.

Proposition 5.7. ([HY12, Proposition 4.6.2]) *Assume that E/F is unramified and let $t \in F^\times$. Then $W_t(s, \phi_0) = 0$ unless $t \in \mathcal{O}_F$. Assume $t \in \mathcal{O}_F$. Recall that $\chi = \chi_{E/F}$ is the quadratic character of $F^\times$ associated to E/F .*

- (1) When $o(t) < o(\beta)$, one has

$$\frac{W_t(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = (1 - q^{-s}) \sum_{0 \leq n \leq o(t)} q^{n(1-s)}.$$

In particular, $W_t(0, \phi_0) = 0$.

- (2) When $o(t) \geq o(\beta)$, and $t \neq 0$, one has

$$\frac{W_t(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = (1 - q^{-s}) \sum_{0 \leq n \leq o(\beta)-1} q^{n(1-s)} + \frac{q^{o(\beta)(1-s)}}{L(s+1, \chi)} \sum_{0 \leq n \leq o(t/\beta)} (\chi(\pi)q^{-s})^n.$$

In particular,

$$L(1, \chi) \frac{W_t(0, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \begin{cases} q^{o(\beta)(o(t/\beta) + 1)} & \text{if } E/F \text{ is split,} \\ q^{o(\beta) \frac{1+(-1)^{o(t/\beta)}}{2}} & \text{if } E/F \text{ is inert.} \end{cases}$$

Moreover, $W_t(0, \phi_0) = 0$ if and only if E/F is inert and $o(t/\beta)$ is odd. When this is the case, one has

$$L(1, \chi) \frac{W'_t(0, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \left[q^{o(\beta) \frac{\text{ord}_\pi(t/\beta) + 1}{2}} + \frac{1 - q^{\text{ord}_\pi(\beta)}}{q(1 - q^{-2})} \right] \log(q).$$

- (3) When $t = 0$, one has

$$\frac{W_0(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = (1 - q^{-s}) \sum_{0 \leq n \leq o(\beta)-1} q^{n(1-s)} + q^{o(\beta)(1-s)} \frac{L(s, \chi)}{L(s+1, \chi)}.$$

Proof. First assume $t \neq 0$. By Proposition [5.1](#) and Lemma [5.2](#)(2), $J_0(0) = 1$ and for $0 < n \leq o(\beta)$,

$$\begin{aligned} J_0(n) &= \int_{\mathcal{O}_F^\times} \psi(-\pi^{-n}bt)db \\ &= \begin{cases} \text{Char}(\pi^n \mathcal{O}_F)(t) - q^{-1} \text{Char}(\pi^{n-1} \mathcal{O}_F)(t) & \text{if } t \neq 0, \\ 1 - q^{-1} & \text{if } t = 0. \end{cases} \end{aligned}$$

and for $n > o(\beta)$,

$$\begin{aligned} J_0(n) &= \int_{\mathcal{O}_F^\times} \chi(b\beta\pi^{-n})|b\beta\pi^{-n}|^{-1}\psi(-\pi^{-n}bt)db \\ &= \begin{cases} \chi(\pi)^{o(\beta)-n}q^{o(\beta)-n}(\text{Char}(\pi^n \mathcal{O}_F)(t) - q^{-1} \text{Char}(\pi^{n-1} \mathcal{O}_F)(t)) & \text{if } t \neq 0, \\ \chi(\pi)^{o(\beta)-n}q^{o(\beta)-n}(1 - q^{-1}) & \text{if } t = 0. \end{cases} \end{aligned}$$

When $t \neq 0$ and $o(t) < o(\beta)$ we have by [\(5.6\)](#),

$$\begin{aligned} \frac{W_t(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} &= 1 + \sum_{0 < n \leq o(\beta)} (1 - q^{-1})q^{n(1-s)} - q^{-1}q^{(o(t)+1)(1-s)} \\ &= (1 - q^{-s}) \sum_{0 \leq n \leq o(t)} q^{n(1-s)}. \end{aligned}$$

When $t \neq 0$ and $o(t) \geq o(\beta)$, we have

$$\begin{aligned} \frac{W_t(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} &= 1 + \sum_{0 < n \leq o(\beta)} (1 - q^{-1})q^{n(1-s)} \\ &\quad + \sum_{o(\beta) < n \leq o(t)} \chi(\pi)^{o(\beta)-n}q^{o(\beta)-n}(1 - q^{-1})q^{n(1-s)} \\ &\quad - q^{-1}\chi(\pi)^{o(\beta)-(o(t)+1)}q^{o(\beta)-(o(t)+1)}q^{(o(t)+1)(1-s)} \\ &= (1 - q^{-s}) \sum_{0 \leq n \leq o(\beta)-1} q^{n(1-s)} + \frac{q^{o(\beta)(1-s)}}{L(s+1, \chi)} \sum_{0 \leq n \leq o(t/\beta)} (\chi(\pi)q^{-s})^n. \end{aligned}$$

When $t = 0$, we have

$$\begin{aligned} \frac{W_t(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} &= 1 + \sum_{0 < n \leq o(\beta)} (1 - q^{-1})q^{n(1-s)} \\ &\quad + \sum_{o(\beta) < n} \chi(\pi)^{o(\beta)-n}q^{o(\beta)-n}(1 - q^{-1})q^{n(1-s)} \\ &= (1 - q^{-s}) \sum_{0 \leq n \leq o(\beta)-1} q^{n(1-s)} + \frac{q^{o(\beta)(1-s)}L(s, \chi)}{L(s+1, \chi)}. \end{aligned}$$

□

Proposition 5.8. ([HY12] Proposition 4.6.3) *Assume that E/F is ramified and let $t \in F$. Then $W_t(s, \phi_0) = 0$ unless $t \in \mathcal{O}_F$. Assume $t \in \mathcal{O}_F$. Recall that $\chi = \chi_{E/F}$ is the quadratic character of $F^\times$ associated to E/F .*

(1) *When $o(t) < o(\beta)$, one has*

$$\frac{W_t(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = (1 - q^{-s}) \sum_{0 \leq n \leq o(t)} q^{n(1-s)}.$$

In particular, $W_t(0, \phi_0) = 0$.

(2) *When $o(t) \geq o(\beta)$, and $t \neq 0$, one has*

$$\frac{W_t(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = |\beta|^{s-1} (1 + \chi(t/\beta) q^{-(o(t/\beta)+f)s}) + (1 - q^{-s}) \sum_{0 \leq n \leq o(\beta)-1} q^{n(1-s)}.$$

In particular,

$$\frac{W_t(0, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = |\beta|^{-1} (1 + \chi(t/\beta)).$$

Moreover, $W_t(0, \phi_0) = 0$ if and only if $\chi(t/\beta) = -1$, i.e., W_β does not represent t . When this is the case,

$$L(1, \chi) \frac{W'_t(0, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = \left[\text{ord}_\pi(t/\beta) + f + \frac{1 - q^{\text{ord}_\pi(\beta)}}{q(1 - q^{-1})} \right] \log(q).$$

(3) *When $t = 0$, one has*

$$\frac{W_0(s, \phi_0)}{\gamma(\beta) \text{vol}(\mathcal{O}_E, d_\beta x)} = |\beta|^{s-1} + (1 - q^{-s}) \sum_{0 \leq n \leq o(\beta)-1} q^{n(1-s)}.$$

5.1. Variants. The first variant is the case when ψ has conductor $n(\psi) = n \neq 0$, where

$$n(\psi) = \{n \in \mathbb{Z} : \psi|_{\pi^n \mathcal{O}_F} = 1\}.$$

Let $\psi'(x) = \psi(ax)$ for some $a \in F$ with $o(a) = n$, then $n(\psi') = 0$, i.e., ψ' is unramified. Let $\beta' = a^{-1}\beta$, then the Weil representation $\omega_{W_\beta, \psi} = \omega_{W_{\beta'}, \psi'}$, and $d_{\psi'} b = |a|^{\frac{1}{2}} d_\psi b$. So

$$\begin{aligned} W_t^\psi(s, \phi) &= \int_F \omega_{W_\beta, \psi}(wn(b))(0) |a(wn(b))|^s \psi(-tb) d_\psi b \\ &= |a|^{-\frac{1}{2}} \int_F \omega_{W_{\beta'}, \psi'}(wn(b))(0) |a(wn(b))|^s \psi(-a^{-1}tb) d_{\psi'} b \\ &= |a|^{-\frac{1}{2}} W_{\frac{t}{a}}^{\psi'}(s, \phi). \end{aligned}$$

Therefore, we have for every $\phi \in S(E)$

$$(5.9) \quad W_t^\psi(s, \phi) = |a|^{-\frac{1}{2}} W_{\frac{t}{a}}^{\psi'}(s, \phi),$$

where the right hand side is with respect to $(W_{\beta'}, \psi')$.

The second variant is the case when $L = \pi_E^c \mathcal{O}_E$ with quadratic form $Q_\beta(x) = \beta x \bar{x}$ (assuming ψ to be unramified). In such a case, let $\tilde{L} = \mathcal{O}_E$ with quadratic form $Q_{\tilde{\beta}}(x) = \tilde{\beta} x \bar{x}$ and $\tilde{\beta} = \pi_E^{2c} \beta$. Then

$$(L, Q_\beta) \cong (\tilde{L}, Q_{\tilde{\beta}}),$$

and

$$W_t(s, \phi_\mu) = W_t(s, \phi_{\tilde{\mu}})$$

with $\phi_\mu = \text{Char}(\mu + L)$ and $\phi_{\tilde{\mu}} = \text{Char}(\tilde{\mu} + \mathcal{O}_E)$.

6. APPENDIX B: COMPUTATION DATA

Theorem [1.4](#) can easily be used to calculate the norm $N(\lambda(\frac{d_1+\sqrt{d_1}}{2}) - \lambda(\frac{d_2+\sqrt{d_2}}{2}))$ up to sign ± 1 . Here are some examples with $\tau_i = \frac{d_i+\sqrt{d_i}}{2}$.

TABLE 7. $d_1 \equiv 5 \pmod{8}$, $d_2 \equiv 5 \pmod{8}$

d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$	d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$
-3	-11	2^{14}	-11	-19	$2^{48}13^6$
-3	-19	$2^{14}3^6$	-11	-35	$2^{96}7^{12}19^6$
-3	-35	$2^{28}5^6$	-19	-35	$2^{96}19^631^641^6$

TABLE 8. $d_1 \equiv 5 \pmod{8}$, $d_2 \equiv 0, 4 \pmod{8}$

d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$	d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$
-3	-4	3	-19	-4	$2^{48}13^6$
-3	-8	5^2	-19	-9	$2^{96}7^{12}19^6$
-3	-20	5^211^2	-19	-20	$2^{96}19^631^641^6$
-11	-4	7^211	-19	-24	$3^{12}13^489^2113^2$
-11	-8	7^413^2	-35	-4	$7^219^231^2$
-11	-20	$11^213^417^419^2$	-35	-8	$5^67^423^461^2$
-11	-24	$13^417^219^441^2$	-35	-24	$19^423^437^441^243^467^489^2$

TABLE 9. $d_1 \equiv 1 \pmod{8}$, $d_2 \equiv 5 \pmod{8}$

d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$	d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$
-7	-3	$3^2 5^2$	-23	-11	$7^{12} 11^4 17^4 19^4 43^2 61^2$
-7	-11	$7^2 13^2 17^2 19^2$	-23	-19	$19^2 37^2 53^2 67^2 79^2 89^2 97^2 103^2 107^2 109^2$
-7	-19	$3^{14} 13^2 31^2$	-23	-35	$5^{18} 7^{12} 19^8 23^4 37^4 43^4 53^4 67^4 181^2 199^2$
-15	-11	$7^8 11^2 13^4 29^2 41^2$	-31	-3	$3^6 11^2 17^2 23^2$
-15	-19	$3^{12} 13^4 29^2 41^2 59^2 71^2$	-31	-11	$11^4 13^6 17^4 29^2 43^2 73^2 79^2 83^2$
-23	-3	$5^6 11^2 17^2$	-31	-19	$3^{40} 13^6 29^4 37^2 127^2$

TABLE 10. $d_1 \equiv 1 \pmod{8}$, $d_2 \equiv 0, 4 \pmod{8}$

d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$	d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$
-7	-4	$2^1 3^2$	-23	-8	$2^6 5^6 7^4 23$
-7	-8	$2^2 5^2 7$	-23	-20	$2^{12} 5^6 11^4 17^2 19^4 53^2$
-7	-20	$2^4 5^2 13^2 17^2$	-23	-24	$2^{12} 17^4 19^2 23^2 37^2 61^2 67^2$
-7	-24	$2^4 3^4 13^2 19^2$	-31	-4	$2^3 3^6 11^2$
-15	-4	$2^2 3^2 7^2$	-31	-8	$2^6 13^2 23^2 29^2 31$
-15	-8	$2^4 5^2 7^2 13^2$	-31	-20	$2^{12} 11^4 13^4 17^2 37^2 53^2 73^2$
-23	-4	$2^3 7^2 11^2$	-31	-24	$2^{12} 3^{12} 13^4 17^4 43^2 61^2$

TABLE 11. $d_1 \equiv 1 \pmod{8}$, $d_2 \equiv 1 \pmod{8}$

d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$	d_1	d_2	$N(\lambda(\tau_1) - \lambda(\tau_2))$
-7	-15	$2^{16} 3^4 5^2$	-15	-23	$2^{48} 5^6 7^8 11^2$
-7	-23	$2^{24} 5^6 7^2$	-15	-31	$2^{48} 3^{12} 11^2 13^4 29^2$
-7	-31	$2^{24} 3^{12} 13^2$	-23	-31	$2^{72} 11^{10} 17^6 37^2 43^2$

Table 6 and Theorem 1.4(1) suggest that $N(\lambda(\frac{d_1+\sqrt{d_1}}{2}) - \lambda(\frac{d_2+\sqrt{d_2}}{2}))$ is a sixth power (up to sign), which can be rephrased as the following conjecture.

Conjecture 6.1. Assume $d_1 \equiv d_2 \equiv 5 \pmod{8}$. For each $t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F$, let $\mathfrak{p}_t$ be the unique prime ideal of F above 2 with $2 \in \mathfrak{p}_t$.

$$\sum_{\substack{t = \frac{m+\sqrt{D}}{2} \in \mathcal{O}_F, \text{ord}_2(m^2-D) \text{ odd} \\ |m| < \sqrt{D}, m \equiv -1 \pmod{4}}} \rho_{E/F}(t\mathfrak{p}_t^{-1})$$

is a multiple of 3.

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